CONTRIBUTIONS TO THE HISTORY OF DETERMINANTS

1900-1920

BY

SIR THOMAS MUIR

D.Sc., LL.D., F.R.S., C.M.G.

Formerly Superintendent-General of Education in Cape Colony

BLACKIE & SON LIMITED

LONDON AND GLASGOW

1930

Printed in Great Britain by Blackie & Son, Ltd., Glasgow

PREFACE

The first instalment (Vol. I, Part 1) of the four-volume *History of Determinants* (Macmillan & Co.; Ltd.) was published in 1890, and in the Preface (pp. v–vii) and Introduction (pp. 3–5) are explained the circumstances which led to it being undertaken and the objects which it sought to attain. Two years afterwards I was induced by Mr. Cecil Rhodes to go out to South Africa to take in hand the organization of Public Education at the Cape; and for a lengthened period this new undertaking was found to be so absorbing that the historical part of my research work had to go to the wall. Strange to say, it was only resumed when the official worries of war-time (1899–1902) made some such mental relief-work a necessity. Progress, however, was now very slow, reference-libraries, for one thing, being 6000 miles away. It thus came about that five years were covered in the preparation of Part 2 of Vol. I, which saw the light in 1906 along with a second edition of Part 1.*

Then followed another series of lean years, the reason being as before. Indeed, so uphill was the struggle that, with the publication of Vol. II in 1911, carrying the record on to 1860, it seemed as if all hope of finishing the work would have to be abandoned. A complete change, however, came about in 1915 with my retirement from the Cape Education Department, and the consequent attainment of freedom to resume the old work in earnest. As a result Vol. III, dealing with the period 1860-1880, was published in 1920 and Vol. IV in 1923, the record of discovery being thus brought down to the end of the nineteenth century in accordance with the plan resolved on forty-two years before.

* See Introduction (pp. 1–5) to this edition.

Of course interest in the subject could not end abruptly with the completion of a set task or the fulfilment of a contract, and so the work of investigation went on without break into the writings of the new century. In a sense the material was ready to hand, as my bibliographical lists had been continued to date: indeed, the eighth, giving the titles up to 1923, had just been published. Fortunately, too, many of the various natural divisions of the subject can be effectively studied without much regard to the rest: and consequently their records can, without appreciable loss, be got ready for publication separately. As a consequence, there have already appeared in the serial publications of Societies, eleven such divisions, the special determinants so dealt with being as a rule those most suitable for such treatment. A list of them will be found below (p. xxiv).

The remaining sections, nearly equal in number and extent, being of more general interest, it was thought better to embody them from the first in book form, and hence the present volume. The only departure from strict adherence to the original scheme consists in using a second time the chapters on Alternants and Compounds. For this accession of strength the volume is indebted to the Council of the R.S.E.

The course thus taken gives me a very welcome opportunity of rendering a service to fellow-workers, who may not be specialists in the study of determinants, by providing as an appendix a Subject-Index of all the historical writings whose story has just been told. Such an Index, I have reason to know, has been a real and growing want since 1924. I have to thank my publishers, Macmillan & Co., Ltd. and Blackie & Son, Ltd., for falling in with this suggestion.

T. M.

RONDEBOSCH, S. AFRICA,
15th August, 1929.

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referred to in the Preface as forming with the present volume a natural extension of the four-volume History.

The titles are shortened for ease in reference. *P.R.S.E.* stands for *Proceedings of the Royal Society of Edinburgh*, and *T.R.S.S.A.* for *Transactions of the Royal Society of South Africa.*

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THE HISTORY OF DETERMINANTS

CHAPTER I

Determinants in general, from 1821 to 1920

The marked activity on which we had to comment when last writing on the subject * we now find to have been more than maintained during the next twenty-year period, the number of writings being at least an eighth more than formerly. One theorem alone is the subject of over a score of contributions, and as a consequence we have, for the convenience of students, segregated the score and made a sort of sub-chapter, I(α), with the heading ' Hadamard's Approximation-theorem '. Almost the same treatment was called for by Schläfli's derived determinant of 1851 (*Hist.*, ii. pp. 52–53). In the case of this there are about a dozen contributions requiring our attention, and these we have less conveniently but justifiably shared between Chapter I and the chapter on Latent Roots. A third instance is the evanescence of an oblong array, a subject which even in the preceding period received exceptional notice from workers without profiting as fully as could have been wished. Indeed in regard to it one feels inclined now to go farther and suggest that a critical study of all that had been written on the subject since Cayley's initial contribution of 1843 (*Hist.*, ii. pp. 14–16) could not fail to be productive of good.

* v. *Theory of Determinants in the Historical Order of Development*, iv. p. 1.

YOUNG, T. (1821)

[Elementary Illustrations of the 'Celestial Mechanics' of Laplace. 344 pp. London.]

When speaking of Drinkwater's paper of 1831 we had to remark on the long period which had elapsed before anything appeared in English on the subject of determinants. Regarding this we have now a slight but interesting correction to make. The first glimmer in the darkness dates ten years before Drinkwater, and was due to the discoverer of the interference of light and of the alphabetical nature of the hieroglyphs on the Rosetta stone.

In the above-noted anonymously-published work there is an Appendix B (pp. 341–4) headed 'Of Interpolation and Extermination', and opening with the theorem: If there be any number of linear equations involving as many unknown quantities in the form

$$\begin{gathered} a_1x + b_1y + \ldots\ldots = A_1, \\ a_2x + b_2y + \ldots\ldots = A_2, \\ \cdots\cdots\cdots\cdots \end{gathered}$$

we shall have

$$x = \frac{\alpha A_1 - \beta A_2 + \gamma A_3 - \ldots\ldots}{\alpha a_1 - \beta a_2 + \gamma a_3 - \ldots\ldots};$$

the coefficients α, β, γ, ... being obtained from the original coefficients by exterminating all the unknown quantities except x in succession. This is followed by two pages of elucidation, in which appear the actual expansions of determinants of the second and third order.

BOTTO, F. S. (1826)

[Memoria sull' eliminazione, contenente una dimostrazione generale della regola di Cramer, . . . 30 pp. Genova.]

The historical position of this overlooked memoir is immediately after that of Schweins' *Producte mit Versetzungen* of 1825, which, it may be remembered, had also the misfortune

to be long neglected (*Hist.*, iv. p. 24). Botto, like Schweins, was capable and well informed, the relevant writings of Euler, Cramer, Bezout, Laplace, and Hindenburg being apparently well known to both. Their two contributions, however, are very unequal in importance, the scope of Botto's being so much the narrower. After a short introduction (pp. 3–4) he carefully lays his foundation by devoting eight pages to the establishment of Cramer's rule: and he then proceeds to his main undertaking, the exposition (pp. 11–22) of a new 'general law' for forming the resultant of two binary quantics. In regard to this it must suffice to say that its nearest relative is Bezout's so-called 'abridged method', the final form obtained for the resultant being in terms of what we now call 2-line determinants. Eight helpful examples of the application of the law are added (pp. 22–30), the fourth of which, where the two equations are of the 5th degree (*Hist.*, ii. pp. 340–341), it will not be surprising to learn, occupies more than a folding double-quarto page without being quite completed.

WEYR, ED. (1889/3)

[O theorii forem bilinearných. 101 + 1 pp. Praze. Also in German in *Monatshefte f. Math. u. Phys.*, i. pp. 163–236.]

The interest of this memoir for us is not at all confined to the subject of the title, the reason being that the author bases his treatment of bilinear forms on Cayleyan matrices, and in consequence is led not only to refer his reader to the classical papers * of Cayley and Frobenius on this subject but actually to devote his first chapter (pp. 163–168) to a re-exposition of the rules of the calculus (*Hist.*, ii. pp. 85–87: iv. p. 285). And this is not all, for the whole of the first ten chapters (pp. 163–218) deal with nothing but matrices: the eleventh (pp. 218–230) is merely an application, however important, in proof of Weierstrass' theorem of 1868 (*Hist.*, iv. pp. 441–442), and the twelfth and last is a similar contribution to a theorem of Fuchs'. The title of the paper might thus well have been *The theory of matrices*

* *Philos. Transac. R.S.*, cxlviii. (1859) pp. 17–37: *Crelle's Journ.*, lxxxiv (1877) pp. 1–63.

with an application to bilinear forms and an application to linear differential equations.

Of the ten chapters it is the second, third and fourth that bear more directly on determinants. The title of the second (pp. 168–172) is ‘Zusammensetzen von Wertesummen’, the subject being what, in the language of simultaneous linear equations, we may call ‘linearly derived solutions’, that is to say, solutions that are aggregates of multiples of those of an already obtained set. The question of the dependence or independence of solutions thus arises: and, as a solution is viewable as constituting a row, the subject comes readily to be the vanishing or non-vanishing of an oblong array. First, an m-by-n array is considered in which all the m rows are linearly independent, several elementary propositions being arrived at, and next the independence is confined to fewer than m rows in the array.

The title of the third chapter (pp. 172–180) ‘Ueber die Nullität der Matrizen’ calls us back to Sylvester. ‘Nullity’ is defined and adopted for use, not however without referring to the related term ‘rank’; and, in the case of a square array, noting the consequent equality

$$\nu + \rho = n$$

(*Hist.*, iv. p. 17). Simultaneous equations are then considered in which the nullity of the determinant of the set is ν. In this connection Weyr's own paper of 1884 is worth recalling as well as the contributions of his contemporaries Giudice, Gordan, Capelli, Garbieri (*Hist.*, iv. pp. 99-106). Finally he is thus led on to a full discussion of Sylvester's approximative theorem regarding the nullity of a product, accompanied by illustrative cases of products in which the nullity is exactly determinable.

HENSEL, K. (1889)

(See chapter on Latent Roots under Rados, 1900.)

BRILL, A. (1890): JOUBKINE, I. (1890)

[Ueber algebraische Correspondenzen. *Math. Annalen*, xxxvi. pp. 321–360.]

[Expression of the partial differential-coefficient of a determinant (In Russian). *Soc. . . . Sci. Natur. Phys.* (Moscow), iv. pp. 29–30.]

The second section of this (pp. 326–333) deals with a theorem of determinants that is needful for the discussion of the author's main subject, the title of the section being 'Conditions for the vanishing of an oblong array'. With unspecialized elements the result is in considerable part a rediscovery of Rahusen's theorem of 1888 (*Hist.*, iv. pp. 40–41). A comparison of the two would be of most use in the case where the number of columns of the array exceeds the number of rows by more than 1.

HADAMARD, J. (1892)

[Essai sur l'étude des fonctions données par leur développement de Taylor. *Journ. (de Liouville) de Math. . . .*, (4) viii. pp. 101–186.]

In the course of this well-known memoir Hadamard has occasion to bring to light and to establish (pp. 148–153) an important identity connecting two sets of minors of an array with $p+1$ rows and p columns. On examination, however, it will be found to belong to what we have already called (*Hist.*, iv. pp. 18–19) "the old subject of vanishing aggregates of products of pairs of determinants", and in the second place to be of the type that is deducible from a simpler by applying the Law of Extensible Minors. If, for example, we take the case where p is 4 and the array is

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_5 & b_5 & c_5 & d_5 \end{matrix}$$

the identity may be written

$$\begin{aligned} |b_1c_4d_5|\ |a_2b_3c_4d_5| &+ |b_2c_4d_5|\ |a_3b_1c_4d_5| \\ &+ |b_3c_4d_5|\ |a_1b_2c_4d_5| = 0, \end{aligned}$$

which is seen to be merely an extensional of

$$b_1 \mid a_2 b_3 \mid + b_2 \mid a_3 b_1 \mid + b_3 \mid a_1 b_2 \mid = 0.$$

Of early discoverers of such extensionals the most appropriate to recall is Desnanot (*Hist.*, i. pp. 139–145).

MERTENS, F. (1893)

[Ueber ganze Functionen von m Systemen von je n Unbestimmten. *Monatshefte f. Math. u. Phys.*, iv. pp. 193–228, 297–329.]

This long paper is at one and the same time a summing-up and a continuation of the author's previous papers on the same subject (*Hist.*, iv. p. 31). In addition, however, to its general interest as treating of functions allied to determinants it touches at several points on determinants themselves. In § 16 (pp. 309–314), for example, two well-known determinants are used as illustrations of a test for divisibility, and are thereby instructively evaluated, the k^{th} compound of an n-line determinant A being shown equal to $A^{(n-1)_{k-1}}$, and Zehfuss' $(m+n)$-line determinant equal to $A^m B^n$.

ANDRÉ, D. (1895): FOLDBERG, P. T. (1896): RADOS, G. ($1898^{17}/_{10}$).

[Mémoire sur les permutations quasi-alternées. *Journ.* (*de Liouville*) *de Math.* . . ., (5) i. pp. 315–350.]

[Determinants Oplösning i Underdeterminanter. *Nyt Tidsskrift f. Mat.*, A vii. pp. 71–72.]

Notwithstanding its title this paper of André's, like others of earlier date on permutations by the same author, contains no reference to the subject of determinants.

The second paper gives a helpful proof for learners.

Radòs' paper is dealt with in chapter on Orthogonants.

KANTOR, S. (1900$^{13}/_1$)

[Ein Theorem über Determinanten. *Nachrichten . . . Ges. d. Wiss.* (Göttingen), 1899, pp. 272–281.]

In 1897 the author in a long geometrical paper * had arrived at a theorem (p. 89) which, when translated into algebraical language, made him take exception to a theorem of Frobenius' of 1876 (*Hist.*, iii. pp. 275–277). Second thoughts on the matter supervened, Frobenius was found justified, the investigation so begun was continued, and the outcome was the paper now reached. The reasoning employed is of the same character as before, and six or seven theorems are formulated. The first is that *all the* m-*line minors of a determinant must vanish if the* m-*line and* (m + 1)-*line coaxial minors vanish and none of the* (m − 1)-*line coaxial minors vanish.* The next four concern axisymmetric and zero-axial skew determinants. The sixth, which resembles the first, is of less interest; and the same may be said of the reference to Kronecker's compound determinant of 1869 in view of other facts already made known (*Hist.*, iv. p. 217).

KÜRSCHÁK, J. (1900$^{22}/_1$)

[Ueber den Rang der Determinante bei inducierten linearen Substitutionen. *Math. u. naturw. Berichte aus Ungarn*, xviii. pp. 229–230.]

In connexion with this paper, a fact which ought to be noticed is that 'the determinant of the induced substitution' is identical with Schläfli's special power-determinant of 1851. The theorem which Kürschák here bases on the theory of linear substitutions must thus have a purely determinantal aspect; and, when we so view it, the statement takes the form: *If* Δ *be a determinant of non-zero rank* k, *then the Schläfli derived determinant whose elements are of the* mth *degree in the elements of* Δ *is of non-*

*Theorie der linearen Strahlencomplexe im Raume von *r* Dimensionen. *Crelle's Journ.*, cxviii. pp. 74–122.

zero rank $(k+m-1)_m$: for example, if $|a_1b_2c_3|$ be of rank 2, the derived determinant

$$\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 & 2a_2a_3 & 2a_3a_1 & 2a_1a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

is of rank $(3)_2$, i.e. 3. A proof in keeping with the view here taken is a desideratum.

MOORE, E. H. ($1900/_2$)

[A fundamental remark concerning determinantal notation, with the evaluation of an important determinant of special form. *Annals of Math.*, (2) i. pp. 177–188.]

The essential subject of this paper is Metzler's important theorem of the preceding year regarding a special determinant with elements that are each the product of k factors (*Hist.*, iv. pp. 495–496). Moore's treatment of it is conspicuously full. His first enunciation of it is: *The product of* n *determinants each of order* m *and* m *determinants each of order* n *may be expressed as a determinant of order* mn *each of whose elements is the product of two factors.* He then skilfully devises a notation by means of which the determinants in the enunciation can be definitely specified, thus making it possible in formulating the theorem to begin with the mn-line determinant and end with its factors—in other words, to make the theorem assume the form of a result in factorisation. With this notation, too, the quite general theorem is as easily stated as the case of it where k is 2. Lastly he gives four distinct proofs—an instructive quartet.

STEPHANOS, C. ($1900/_{1-3}$)

[Sur une extension du calcul des substitutions linéaires. *Journ. (de Liouville) de Math.* . . ., (5) vi. pp. 73–128.]

This is the full memoir of which we have already had foretastes in 1898 and 1899 (*Hist.*, iv. pp. 72, 74). So far as determinants are concerned there is little additional to be drawn attention to, save the paragraphs that deal with the linear transformation of the minors of an oblong array (§§ 16, 13, . . .).

MUIR, T. (1900/3)

[On certain aggregates of determinant minors. *Proceed. R. Soc. Edinburgh*, xxiii, pp. 142-154.]

The minors in question belong to any even-ordered determinant, say

$$\begin{vmatrix} 1 & 2 & 3 & \dots & 2m \\ 1 & 2 & 3 & \dots & 2m \end{vmatrix}.$$

The typical one of the first aggregate is

$$\begin{vmatrix} 1 & 2 & \dots & m \\ m+1 & m+2 & \dots & 2m \end{vmatrix}$$

the others being got from it by changing the last of the upper indices with each of the lower in succession. The first result is obtained by expanding each minor of the aggregate in terms of the elements of the last row and their cofactors, and then combining the terms in pairs, the typical combined term being

$$\begin{vmatrix} 1 & 2 & \dots & m-1 \\ m+1 & m+2 & \dots & 2m-1 \end{vmatrix} \begin{pmatrix} m & 2m \\ 2m & m \end{pmatrix}.$$

The cases where m is 3 and 4 are worked out, and then the general equality is carefully formulated. When m is 3 the result is

$$\begin{vmatrix} 123 \\ 456 \end{vmatrix} - \begin{vmatrix} 124 \\ 356 \end{vmatrix} + \begin{vmatrix} 125 \\ 346 \end{vmatrix} - \begin{vmatrix} 126 \\ 345 \end{vmatrix}$$

$$= \begin{vmatrix} 12 \\ 45 \end{vmatrix} \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix} - \begin{vmatrix} 12 \\ 46 \end{vmatrix} \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} + \begin{vmatrix} 12 \\ 56 \end{vmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

$$- \begin{vmatrix} 12 \\ 35 \end{vmatrix} \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} + \begin{vmatrix} 12 \\ 36 \end{vmatrix} \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$$

$$+ \begin{vmatrix} 12 \\ 34 \end{vmatrix} \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix};$$

and when in addition $\begin{vmatrix} 3456 \\ 3456 \end{vmatrix}$ is axisymmetric the right-hand side vanishes, and the outcome is Kronecker's 'linear relation' of 1882 (*Hist.*, iv. p. 113).

In the next aggregate there is a typical *pair* of minors, the

members of a pair being conjugates. When m is 3 the equality is

$$\begin{vmatrix}451\\456\end{vmatrix} - \begin{vmatrix}456\\451\end{vmatrix} + \begin{vmatrix}426\\456\end{vmatrix} - \begin{vmatrix}456\\426\end{vmatrix} + \begin{vmatrix}356\\456\end{vmatrix} - \begin{vmatrix}456\\356\end{vmatrix}$$

$$= \begin{vmatrix} \genfrac{}{}{0pt}{}{4}{4} & \genfrac{}{}{0pt}{}{4}{5} & \genfrac{}{}{0pt}{}{4}{6} \\[6pt] \genfrac{}{}{0pt}{}{5}{4} & \genfrac{}{}{0pt}{}{5}{5} & \genfrac{}{}{0pt}{}{5}{6} \\[6pt] \genfrac{}{}{0pt}{}{1}{4}-\genfrac{}{}{0pt}{}{6}{3} & \genfrac{}{}{0pt}{}{1}{5}-\genfrac{}{}{0pt}{}{6}{2} & \genfrac{}{}{0pt}{}{1}{6}-\genfrac{}{}{0pt}{}{6}{1} \end{vmatrix} + \begin{vmatrix} \genfrac{}{}{0pt}{}{4}{4} & \genfrac{}{}{0pt}{}{4}{5} & \genfrac{}{}{0pt}{}{4}{6} \\[6pt] \genfrac{}{}{0pt}{}{2}{4}-\genfrac{}{}{0pt}{}{5}{3} & \genfrac{}{}{0pt}{}{2}{5}-\genfrac{}{}{0pt}{}{5}{2} & \genfrac{}{}{0pt}{}{2}{6}-\genfrac{}{}{0pt}{}{5}{1} \\[6pt] \genfrac{}{}{0pt}{}{6}{4} & \genfrac{}{}{0pt}{}{6}{5} & \genfrac{}{}{0pt}{}{6}{6} \end{vmatrix} + \begin{vmatrix} \genfrac{}{}{0pt}{}{3}{4}-\genfrac{}{}{0pt}{}{4}{3} & \genfrac{}{}{0pt}{}{3}{5}-\genfrac{}{}{0pt}{}{4}{2} & \genfrac{}{}{0pt}{}{3}{6}-\genfrac{}{}{0pt}{}{4}{1} \\[6pt] \genfrac{}{}{0pt}{}{5}{4} & \genfrac{}{}{0pt}{}{5}{5} & \genfrac{}{}{0pt}{}{5}{6} \\[6pt] \genfrac{}{}{0pt}{}{6}{4} & \genfrac{}{}{0pt}{}{6}{5} & \genfrac{}{}{0pt}{}{6}{6} \end{vmatrix}$$

where this time the right-hand side vanishes when the given six-line determinant is centro-symmetric.

TRAVERSO, N. (1900/$_9$)

[Sopra una generalizzazione della teoria dei determinanti. *Giornale di Mat.*, xxxix. pp. 225–239: xl. pp. 308–324.]

A rough conception of the nature of the generalization in question may be obtained from two facts: (1) The variables of the new function are arranged in rows, but the rows are not restricted to be of the same length: (2) the terms of the function involve a constant number of variables, but the number to be taken from a row is not restricted to one.

The exposition would have been much lightened by simple concrete illustrations; and would have been more attractive had any indication been given of applicability towards the solution of existing algebraical problems.

MACLOSKIE, G. (1900/$_1$, 1904/$_2$): CADENAT, A. (1900)

[A method of solving determinants. *Annals of Math.*, (2) i. pp. 74–76.]

[A general method of evaluating determinants. *Annals of Math.*, (2) vi. p. 30.]

[Règle pratique pour obtenir le développement d'un déterminant de degré quelconque. *Assoc. franç. pour l'avancem. des sci.* (Paris), pp. 241–247.]

The property whose rediscovery is here made known in the *Annals* is Hermite's of 1849 or Chio's of 1853 (*Hist.*, ii. pp. 46, 79–81): and the other writer's 'practical rule' for forming the terms directly from the individual elements is not more useful than quite a number of others already brought forward with the same object in view (e.g. *Hist.*, iv. pp. 17-18, . . ., 85–86).

BILENKI, H. ($1900/_5$)

[Note sur les permutants. *Nouv. Annales de Math.*, (3) xix. pp. 213–216.]

The 'permutants' here referred to are not those of Cayley and Sylvester (*Hist.*, ii. pp. 63–68, 266–267): nor are they claimed to have any connection with the original permutants or with determinants.

CAZZANIGA, T. ($1900^4/_5$)

[Qualche complemento al teorema di Hunyady su certi determinanti. *Periodico di Mat.*, (2) iii. pp. 17–22.]

The theorem here called Hunyady's on the authority of E. Pascal is the simplest case of Schläfli's of 1851, namely, that in which the elements of the determinant are of the 2nd degree. The fresh contribution now made concerns the primary minors of the said determinant, which are all shown to have for a factor the $(n-1)^{\text{th}}$ power of the basic determinant, the cofactors also being given. The fundamental part of the procedure consists in taking the basic determinant as the determinant of a set of linear equations in $x_1, x_2, \ldots, x_n$, and the Schläfli determinant as the determinant of the derived set of equations in $x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots, x_{n-1}x_n$, and solving both sets of equations. It is well to note, however, that this, while effective enough for the case under consideration, would have less to commend it with elements of a higher degree than the 2nd. As a simple deduction the evaluation of the determinant of the primary minors is next effected: and then, finally, the results reached are applied to two special determinants—the orthogonant and the zero-axial skew determinant of even order.

A concluding theorem summing up the separate facts arrived

at would seem a natural addendum. This for the case where n is 3 would be: *the basic determinant being* $|a_1b_2c_3|$, *the adjugate of Schläfli's derived determinant*

$$\begin{vmatrix} a_1^2 & 2a_1b_1 & 2a_1c_1 & b_1^2 & 2b_1c_1 & c_1^2 \\ a_1b_1 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & b_1b_2 & b_1c_2+b_2c_1 & c_1c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

is equal to $|A_1B_2C_3|^{2\cdot 6} \times$

$$\begin{vmatrix} A_1^2 & 2A_1B_1 & 2A_1C_1 & B_1^2 & 2B_1C_1 & C_1^2 \\ A_1B_1 & A_1B_2+A_2B_1 & A_1C_2+A_2C_1 & B_1B_2 & B_1C_2+B_2C_1 & C_1C_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

where $|A_1B_2C_3|$ *is the adjugate of* $|a_1b_2c_3|$.

GAVRILOVITCH, B. ($1900^{15}/_5$)

[On a property of determinants (In Serbian). *Glasa Srpske Kral. Akad.*, lxiii. pp. 115–130.]

The property in question we may condense for ourselves as follows: *If the elements of the principal diagonal of an n-line determinant be changed in sign, and likewise the elements of every alternate diagonal parallel to the principal, the determinant is in effect multiplied by* $(-1)^n$: *if, on the other hand, the change be not made in these diagonals but in all the others parallel to the principal the determinant remains unaltered in value.* The latter portion is identical with Janni's of 1874 (*Hist.*, iii. pp. 51, 82): and the former is derived from the latter by multiplying all the columns by -1. Illustrative examples are taken from persymmetric recurrents.

SAINTE-MARIE, C. F. ($1900^1/_9$)

[Mineurs d'un déterminant. *L'Intermédiaire des Math.*, vii. pp. 326, 416–420.]

Another rediscovery, with a laboured proof, of Sylvester's theorem of 1851 regarding an aggregate of products of pairs of determinants. Only about a year had elapsed since a previous rediscovery (*Hist.*, iv. p. 75).

JÜRGENS, E. ($1900^{16}/_9$)

[Numerische Berechnung von Determinanten. *Jahresb. d. deutschen Math.-Verein.*, ix. (1) pp. 131–136.]

Although the title given by the author here is only half of the title of his paper of 1886 (*Hist.*, iv. pp. 101–102), the contents of the two papers are essentially the same: and this, the shorter of the two, shows no improvement on the other.

CARLINI, L. ($1900/_{12}$), ($1901/_2$)

[Sul prodotto di due matrici rettangolari conjugate. *Periodico di Mat.*, (2) iii. pp. 193–198.]

[Sopra due tipi di relazioni fra i prodotti delle coppie di matrici conjugati formati coi medesimi elementi. *Periodico di Mat.*, (2) iv. pp. 175–179.]

The really fresh matter brought forward here is a proposal to give a meaning to the product of two oblong arrays that differ from each other in length and breadth but are alike in excess of breadth over length: for example, a meaning for the symbol

$$\left\| \begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \right\| \cdot \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{matrix} \right\|.$$

Clearly such a meaning must be in keeping with the meaning already accepted for

$$\left\| \begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \right\| \cdot \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\|,$$

and must have associated with it a multiplication-theorem analogous to

$$\begin{vmatrix} x_1a_1 + x_2a_2 + x_3a_3 & x_1b_1 + x_2b_2 + x_3b_3 \\ y_1a_1 + y_2a_2 + y_3a_3 & y_1b_1 + y_2b_2 + y_3b_3 \end{vmatrix}$$
$$= |x_1y_2|\ |a_1b_2| + |x_1y_3|\ |a_1b_3| + |x_2y_3|\ |a_2b_3|.$$

Further, if with this in view each term of the expanded equivalent of the symbol is to be the product of a minor from the first array by a minor from the second, then the difficulty must be faced that in the second array there is a superfluity of minors which makes pairing impossible without the previous imposition of a principle of choice. All this the author has seen, but it is very doubtful whether he has taken the best way to make his proposal acceptable. His plan is to have for each column of the first array a column of the second array prearranged as being 'corresponding' to it: for example, he says "Agreeing to indicate a pair of corresponding columns by giving them the same second suffix, we have

$$\left\|\begin{matrix} a_{12} & a_{14} & a_{15} \\ a_{22} & a_{24} & a_{25} \end{matrix}\right\| \cdot \left\|\begin{matrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \end{matrix}\right\| = \left|\begin{matrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{matrix}\right| \cdot \left| b_{11}\, b_{22}\, b_{33}\, b_{44} \right| + \left|\begin{matrix} a_{12} & a_{15} \\ a_{22} & a_{25} \end{matrix}\right| \cdot \left| b_{11}\, b_{22}\, b_{33}\, b_{45} \right| + \left|\begin{matrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{matrix}\right| \cdot \left| b_{11}\, b_{23}\, b_{34}\, b_{45} \right|\text{"};$$

and he carefully goes on to show that the single determinant which equals this sum of products is

$$\left|\begin{matrix} b_{11} & a_{12}b_{12} + a_{14}b_{14} + a_{15}b_{15} & b_{13} & a_{22}b_{12} + a_{24}b_{14} + a_{25}b_{15} \\ b_{21} & a_{12}b_{22} + a_{14}b_{24} + a_{15}b_{25} & b_{23} & a_{22}b_{22} + a_{24}b_{24} + a_{25}b_{25} \\ b_{31} & a_{12}b_{32} + a_{14}b_{34} + a_{15}b_{35} & b_{33} & a_{22}b_{32} + a_{24}b_{34} + a_{25}b_{35} \\ b_{41} & a_{12}b_{42} + a_{14}b_{44} + a_{15}b_{45} & b_{43} & a_{22}b_{42} + a_{24}b_{44} + a_{25}b_{45} \end{matrix}\right|.$$

Fortunately this can be done much more shortly and instructively by taking advantage of the fact that the new theorem is viewable as a case of the old. Thus changing the 2-by-3 array into a 4-by-5 array like its fellow, namely, into

$$\left\|\begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{12} & \cdot & a_{14} & a_{15} \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & a_{22} & \cdot & a_{24} & a_{25} \end{matrix}\right\|$$

we obtain at once by using the old multiplication-theorem

$$\begin{vmatrix} b_{11} & a_{12}b_{12}+a_{14}b_{14}+a_{15}b_{15} & b_{13} & a_{22}b_{12}+a_{24}b_{14}+a_{25}b_{15} \\ b_{21} & a_{12}b_{22}+a_{14}b_{24}+a_{15}b_{25} & b_{23} & a_{22}b_{22}+a_{24}b_{24}+a_{25}b_{25} \\ b_{31} & a_{12}b_{32}+a_{14}b_{34}+a_{15}b_{35} & b_{33} & a_{22}b_{32}+a_{24}b_{34}+a_{25}b_{35} \\ b_{41} & a_{12}b_{42}+a_{14}b_{44}+a_{15}b_{45} & b_{43} & a_{22}b_{42}+a_{24}b_{44}+a_{25}b_{45} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & . & . & . \\ . & a_{12} & . & a_{14} \\ . & . & 1 & . \\ . & a_{22} & . & a_{24} \end{vmatrix} \left| b_{11}b_{22}b_{33}b_{44} \right| + \begin{vmatrix} 1 & . & . & . \\ . & a_{12} & . & a_{15} \\ . & . & 1 & . \\ . & a_{22} & . & a_{25} \end{vmatrix} \left| b_{11}b_{22}b_{33}b_{45} \right| .$$

$$+ \begin{vmatrix} 1 & . & . & . \\ . & . & a_{14} & a_{15} \\ . & 1 & . & . \\ . & . & a_{24} & a_{25} \end{vmatrix} \left| b_{11}b_{23}b_{34}b_{45} \right|$$

$$= \left| a_{12}a_{24} \right| \cdot \left| b_{11}b_{22}b_{33}b_{44} \right| + \left| a_{12}a_{25} \right| \cdot \left| b_{11}b_{22}b_{33}b_{45} \right|$$
$$+ \left| a_{14}a_{25} \right| \cdot \left| b_{11}b_{23}b_{34}b_{45} \right| ,$$

where the want of agreement in the matter of the last sign suggests that the user of the new symbol has another possible pitfall to keep an eye on besides that of ‘ corresponding ’ columns.

The latter half of the paper is concerned with a theorem exemplifying the usefulness of the symbol in actual work on compound determinants.

The second paper originates in there being, as we have seen, a multiplicity of ways of selecting columns of the second array to ‘ correspond ’ to columns of the first array.

METZLER, W. H. (1901/$_3$)

[On certain aggregates of determinant minors. *Transac. American Math. Soc.*, ii. pp. 395-403.]

Metzler here takes up Muir's pair of theorems of 1900, spoken of above and illustrated but not formulated, and extends them still further, the generalization made being that which is rendered possible by passing from elements and primary minors in using Laplace's expansion-theorem to k-line and $(n-k)$-line minors.

The nature of the results reached may thus be readily anticipated in the rough: but the accurate formulation of them is not a brief or simple matter. As an illustrative example of the more interesting theorem of the two we have

$$\begin{vmatrix} 1234 \\ 5678 \end{vmatrix} - \begin{vmatrix} 1235 \\ 4678 \end{vmatrix} + \cdots + \begin{vmatrix} 1278 \\ 3458 \end{vmatrix}$$

$$= \begin{vmatrix} 12 \\ 34 \end{vmatrix} \left\{ \begin{vmatrix} 56 \\ 78 \end{vmatrix} - \begin{vmatrix} 57 \\ 68 \end{vmatrix} + \begin{vmatrix} 58 \\ 67 \end{vmatrix} + \begin{vmatrix} 67 \\ 58 \end{vmatrix} - \begin{vmatrix} 68 \\ 57 \end{vmatrix} + \begin{vmatrix} 78 \\ 56 \end{vmatrix} \right\}$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$+ \begin{vmatrix} 12 \\ 78 \end{vmatrix} \left\{ \begin{vmatrix} 34 \\ 56 \end{vmatrix} - \begin{vmatrix} 35 \\ 46 \end{vmatrix} + \begin{vmatrix} 36 \\ 45 \end{vmatrix} + \begin{vmatrix} 45 \\ 36 \end{vmatrix} - \begin{vmatrix} 46 \\ 35 \end{vmatrix} + \begin{vmatrix} 56 \\ 34 \end{vmatrix} \right\}.$$

PASCAL, E. (1901/$_4$)

[Un semplice teorema relativo alle caratteristiche di certe matrici rettangolari composte mediante altre. *Rendiconti . . . R. Ist. Lombardo*, (2) xxxiv. pp. 539–541.]

The theorem in question we may conveniently express for ourselves as follows: *If an oblong array* O *occur* k *times as a minor in another array* Ω, *and in such circumstances that every row of* Ω *contains a row of* O *and only zeros besides, and every column of* Ω *contains a column of* O *and only zeros besides, then the characteristic* (*or non-zero rank*) *of* Ω *is* k *times the characteristic of* O. For example, the characteristic of

$$\begin{array}{ccccccccc} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 & b_3 \end{array}$$

is three times the characteristic of

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 . \end{array}$$

NANSON, E. J. (1901/6, 1902/9): BES, K. (1903)

[Questions 14899, 15190. *Educ. Times*, liv. p. 262: lv. p. 396.]

[La dépendance ou l'indépendance d'un système d'équations algébriques. *Verhand. . . . Akad. . . .* (*Amsterdam*), viii. (1) no. 6, 29 pp.]

The subject which these two questions of Nanson's concern is that which originated with Cayley in 1843 (*Hist.*, ii. pp. 14–16) and which we have last seen dealt with in 1897 (*Hist.*, iv. pp. 61, 67). The first requires proof that all the p-line determinants of a p-by-q array can be expressed in terms of $p(q - p) + 1$ of them properly chosen: and the second that the array

$$\left\| \begin{matrix} A & B & C & F & G & H \\ A' & B' & C' & F' & G' & H' \\ A'' & B'' & C'' & F'' & G'' & H'' \end{matrix} \right\|$$

where

$$A = \begin{vmatrix} b & f \\ f & c \end{vmatrix}, \quad A' = \begin{vmatrix} b' & f' \\ f' & c' \end{vmatrix}, \quad A'' = \begin{vmatrix} b & f' \\ f & c' \end{vmatrix} + \begin{vmatrix} b' & f \\ f' & c \end{vmatrix}, \ \ldots$$

needs for evanescence only two relations.

Bes' otherwise interesting paper discusses the same subject in an appendix (pp. 24–26): it however carries us little further than the point reached by Cayley.

LAISANT, C. A. (1901 23/7)

[Aufgabe 17. *Archiv d. Math. u. Phys.*, (3) i. p. 370: xiii. pp. 191–192.]

The theorem here established is that *If in an* n-*line determinant the diagonals below and parallel to the main diagonal be respectively multiplied by*

$$\begin{array}{c} x_1,\ x_2,\ \ \ldots,\ \ x_{n-1} \\ x_1x_2,\ x_2x_3,\ \ \ldots,\ \ x_{n-2}\,x_{n-1} \\ x_1x_2x_3,\ x_2x_3x_4,\ \ \ldots,\ \ x_{n-3}\,x_{n-2}\,x_{n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_1x_2x_3\ \ldots\ x_{n-1}, \end{array}$$

and the diagonals above the main diagonal be divided respectively by the same, the determinant is unaltered in value. We note that the case of it where all the x's are equal is Fürstenau's of 1879 (*Hist.*, iii. p. 82). It will be found generalized by us in Chapter VIII under Pascal, E. ($1907/_2$).

SCHUR, I. (1901)

[Ueber eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen. *Inaug. Dissert.*, 75 pp. Berlin.]

The first remark which it is necessary to make about this so-called class of matrices is that the matrices in question are all of them square, and that thus the subject of determinants is even a shade nearer to the reader of the dissertation than might at first appear. Besides if there be a class of square matrices worthy of separate study, the great probability is that there is also a class of determinants calling for some little attention. In either case our first aim must be to indicate the characteristics of the class. It being understood, then, that the elements of $|a_{rs}|$ and $|b_{rs}|$ are independent variables, and that the c's of $|c_{rs}|$ are defined in terms of the a's and b's by the identity

$$|a_{rs}| \cdot |b_{rs}| \equiv |c_{rs}|,$$

it is held that if D_a be a determinant of any order whose elements are rational integral functions of the elements of $|a_{rs}|$, if D_b, D_c be the equigrade determinants formed in the same way from $|b_{rs}|$, $|c_{rs}|$ respectively, and if D be such that $D_a \cdot D_b = D_c$, then D_a belongs to the class in question. Evidently, recognition of a determinant as belonging to the class does not at once result on superficial inspection. Following the lead of Frobenius the author investigates the properties of the class and expounds them through the length of eight chapters, the results being summed in twenty-eight carefully formulated theorems. It only seems unfortunate that in seeking for a word to designate the new class and at the same time partially to indicate the nature of the relation between the basic and the derived determinant he should have made choice of one already so fully appropriated and otherwise so unsuitable as *invariant*. To call the above

determinant D_a an invariant form of $|a_{rs}|$ is clearly far from helpful in its suggestiveness. One of the theorems we must note as being of special interest to us, having already made an appearance in different surroundings under Deruyts in 1896 (*Hist.*, iv. p. 64). It may be put shortly thus: *If* $|a_{rs}|$ *be* n-*line*, D_a *be* p-*line*, *and the elements of* D_a *be all of the* q^{th} *dimension in the* a*'s*, *then*

$$D_a = |a_{rs}|^{pq \div n}.$$

Schläfli's theorem of 1851, we may also note, is viewable as a particular case of it (*Hist.*, ii. pp. 52–53.)

FREDHOLM, J. ($1902^{27}/_1$)

[Sur une classe de transformations rationelles. Sur une classe d'équations fonctionnelles. *Comptes rendus . . . Acad. des Sci.* (Paris), cxxxiv. pp. 219–222, 1561–1564.]

The real as well as the ostensible subject of these two notes is not general determinants, and the inclusion of them in the present chapter may therefore seem unwarranted. Nevertheless it is desirable, for the sake of breadth of view, to draw passing attention to them; for it will be found later under a different heading that the theory of his equation, as the author himself says, is but a limiting case of the theory of linear equations, and that in the development of it one finds anew 'all the results of the theory of determinants'.

MEYER, W. F.: LOEWY, A.: ETC. (1901–1905)

[Sprechsaal für die 'Encyk. d. math. Wissen.' *Archiv d. Math. u. Phys.*, (3) ii. pp. 230, 232: iii. pp. 86–173: v. p. 341: ix. p. 103.]

Corrections, amendments and additions, all of a minor character, relating to the subject of determinants as treated in the 'Encyklopädie' (*Hist.*, iv. p. 95).

NEWSON, H. B. ($1902^2/_1$)

[Note on the product of linear substitutions. *Annals of Math.*, (2) iii. pp. 147–148.]

For the case of three variables the result here reached is that if

$$\begin{array}{lcl} \rho x_1 = a_1x + b_1y + c_1z & & \rho_1x_2 = \alpha_1x + \beta_1y + \gamma_1z \\ \rho y_1 = a_2x + b_2y + c_2z & \text{and} & \rho_1y_2 = \alpha_2x + \beta_2y + \gamma_2z \\ \rho z_1 = a_3x + b_3y + c_3z & & \rho_1z_2 = \alpha_3x + \beta_3y + \gamma_3z \end{array}$$

then

$$\rho\rho_1x_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ \cdot & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} \div |a_1b_2c_3|, \quad \ldots\ldots\ldots$$

In regard to this we note for ourselves that the ordinary simple expression for $\rho\rho_1x_2$ may be got from it by using the theorem concerning the product of a bilinear form and its discriminant, namely:

$$\begin{array}{ccc|c} x & y & z & \\ \hline a_1 & b_1 & c_1 & \xi \\ a_2 & b_2 & c_2 & \eta \\ a_3 & b_3 & c_3 & \zeta \end{array} \cdot |a_1b_2c_3| = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ \cdot & \xi & \eta & \zeta \end{vmatrix};$$

and we recall Sylvester's paper of 1850 (*Hist.*, ii. pp. 117–119.)

METZLER, W. H. ($1902^{27}/_5$)

[On a theorem regarding determinants with polynomial elements. *Transac. R. Soc. Canada*, (2) viii. section 3, pp. 157–160.]

This paper follows up Muir's of 1883 on the same subject (*Hist.*, iv. pp. 22–23), the object being to find what

$$D_{n,p} - \Sigma D_{n,p-1} + \Sigma D_{n,p-2} - \ldots + (-1)^{p-1}D_{n,1}$$

reduces to when p is not greater than n. The procedure consists

in transforming $D_{n,p}$, $\Sigma D_{n,p-1}$, . . . into sums of determinants with monomial elements, and then making the necessary cancellations; the result, consequently, takes the form of a sum of such determinants: and an expression is obtained for it. For example:

$$D_{3,3}-\Sigma D_{3,2}+\Sigma D_{3,1}$$

$$\text{i.e.}\quad \begin{vmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} & a_{13}+b_{13}+c_{13} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} & a_{23}+b_{23}+c_{23} \\ a_{31}+b_{31}+c_{31} & a_{32}+b_{32}+c_{32} & a_{33}+b_{33}+c_{33} \end{vmatrix}$$

$$-\Sigma \begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{vmatrix} + \Sigma \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{11} & b_{12} & b_{13} \\ c_{11} & c_{12} & c_{13} \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}.$$

NICOLETTI, O. ($1902^{12}/_{6}$)

[Sulle matrici associate ad una matrici data. *Atti . . . Accad. . . .* (Torino), xxxvii. pp. 463–467 (or pp. 655-659.]

The arrays (or matrices) in question are simply the *compounds* of the given array. When, therefore, the latter is 4-by-5, its second compound is 6-by-10, its third compound 4-by-10, and its fourth 1-by-5. The properties brought forward mainly concern the 'characteristic' or 'rank'. The first is: *If an array be of non-zero rank* r, *then the rank of the* p^{th} *compound is* $C_{r,p}$*. This implies for one thing, that when $p=r$ the elements of any two rows of the compound are proportional. For the other and less important properties a modification of the idea of rank is introduced under the designation 'rank with regard to a certain minor', the meaning being that, in determining this rank of an array, only majors of that particular minor are to be taken into account.

* See under Orlando (1902) in Chap. VII.

TRAVERSO, N. (1902/7)

[Sulle principali operazioni dell' analisi combinatoria formale e su alcuni loro applicazioni . . . *Periodico di Mat.*, (2) v. pp. 1–30, 73–116, 153–184.]

This disquisition of 106 pages requires a passing notice for two reasons: in the first place because it supplies illustrative examples awanting from the author's paper of 1900, and in the second place because of a section (pp. 27–30) bearing the heading 'Methods for the rapid development of determinants'. These latter, unfortunately, are only sure and speedy ways of obtaining the permutations of 1, 2, 3, . . ., n, and therefore hardly to be recommended when the elements of the determinant are, as in the example given, a mixture of positive and negative integers.

MUIR, T. (1902 8/9): MIREA, St. N. (1902): MUIR, T. (1904 25/7)

[The generating functions of certain special determinants. *Proceed. R. Soc. Edinburgh*, xxiv. pp. 387–392.]

[O proprietate a determinanților. *Gazeta Math.* (Bucuresti), vii. pp. 245–248.]

[The sum of the signed primary minors of a determinant. *Proceed. R. Soc. Edinburgh*, xxv. pp. 372–382.]

Incidentally there are here formulated (pp. 390–391) six simple theorems connected with the sum of the signed primary minors of a determinant. One of them was found afterwards by the author to be essentially due to Battaglini and another to Routh (*Hist.*, iii. pp. 11, 13): and of the others the following may be worth mentioning: If the elements of a determinant be all increased by the same quantity ω, the determinant is thereby increased by ω times the sum of the signed primary minors: for example:

$$\begin{vmatrix} a_1+\omega & a_2+\omega & a_3+\omega \\ b_1+\omega & b_2+\omega & b_3+\omega \\ c_1+\omega & c_2+\omega & c_3+\omega \end{vmatrix} = |\,a_1b_2c_3\,| - \omega \begin{vmatrix} \cdot & 1 & 1 & 1 \\ 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \end{vmatrix}.$$

Mirea's property, which appeared two months earlier, is what is suggested by the result of putting in Muir's theorem the sum of the signed primary minors equal to 0.

The third paper, Muir's of 1904, contributes nothing fresh except under continuants.

MUIR, T. ($1902^8/_9$)

[The generating function of the reciprocal of a determinant. *Transac. R. Soc. Edinburgh*, xl. pp. 615–629.]

The initial object here is to supply a proof, long wanted, of Jacobi's theorem of 1829 (*Hist.*, i. pp. 188–189) that *the coefficient of* $x_1^{-1}x_2^{-1}x_3^{-1}\ldots$ *in the expansion of*

$$(a_1x_1 + a_2x_2 + a_3x_3 + \ldots)^{-1}(b_2x_2 + b_3x_3 + \ldots + b_1x_1)^{-1} \times (c_3x_3 + \ldots + c_1x_1 + c_2x_2)^{-1}$$

is
$$|\,a_1b_2c_3\ldots\,|^{-1}.$$

Jacobi's attempt is first duly discussed (pp. 615–619) with a view to make clear the difficulties inherent in his procedure. The proposed substitute is then illustrated by the case where the number of linear functions is 4. The starting point is the identity

$$0 = \begin{vmatrix} U_1 - a_2x_2 & -b_2x_2 & -c_2x_2 & -d_2x_2 \\ -a_3x_3 & U_2 - b_3x_3 & -c_3x_3 & -d_3x_3 \\ -a_4x_4 & -b_4x_4 & U_3 - c_4x_4 & -d_4x_4 \\ -a_1x_1 & -b_1x_1 & -c_1x_1 & U_4 - d_1x_1 \end{vmatrix}$$

due to the vanishing of the sum of every column. Expanding the determinant in a series arranged according to products of the U's we have

$$0 = U_1U_2U_3U_4 - \Sigma U_1U_2U_3\cdot d_1x_1 + \Sigma U_1U_2\cdot|\,c_4d_1\,|\cdot x_4x_1 + \Sigma U_1\cdot|\,b_3c_4d_1\,|\cdot x_3x_4x_1 + |\,a_2b_3c_4d_1\,|\cdot x_2x_3x_4x_1,$$

and consequently

$$|\,a_1b_2c_3d_4\,|\cdot x_1x_2x_3x_4 = U_1U_2U_3U_4 - \mathring{\Sigma} U_1U_2U_3\cdot d_1x_1 + \mathring{\Sigma} U_1U_2\cdot|\,c_4d_1\,|\cdot x_4x_1 + \mathring{\Sigma} U_1U_3\cdot|\,b_3d_1\,|\cdot x_3x_1 - \mathring{\Sigma} U_1\cdot|\,b_3c_4d_1\,|\cdot x_3x_4x_1$$

where the circlet surmounting the sign of summation denotes cyclical change of suffixes. On dividing by $x_1x_2x_3x_4 \cdot U_1U_2U_3U_4$ this is changed into

$$\frac{|a_1b_2c_3d_4|}{U_1U_2U_3U_4} = \frac{1}{x_1x_2x_3x_4} - \overset{\circ}{\Sigma}\frac{d_1}{U_4x_2x_3x_4} + \overset{\circ}{\Sigma}\frac{|c_4d_1|}{U_3U_4x_2x_3}$$
$$+ \overset{\circ}{\Sigma}\frac{|b_3d_1|}{U_2U_4x_2x_4} - \overset{\circ}{\Sigma}\frac{|b_3c_4d_1|}{U_2U_3U_4x_1};$$

and as the expansion of each of the four fractions under the sign of summation contains in every term a positive power of x, it follows that the cofactor of $x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}$ in the expansion of $|a_1b_2c_3d_4|/U_1U_2U_3U_4$ is 1, as was to be proved. A first generalization is then made, which involves also MacMahon's main theorem of 1893 (*Hist.*, iv. pp. 485–486): and finally this is widened into the following: *In the expansion of*

$$\{m_1 - \tau_1(U_1 - \omega_1)\}^{-1} \cdot \{m_2 - \tau_2(U_2 - \omega_2)\}^{-1} \cdots\cdots$$

the aggregate of the terms which are such that every τ occurring in them is accompanied by the corresponding $\mathrm{x} - \xi$, *and vice versa, is*

$$\begin{vmatrix} m_1 - a_1\tau_1(x_1 - \xi_1) & - b_1\tau_1(x_1 - \xi_1) & - c_1\tau_1(x_1 - \xi_1) & \cdots \\ - a_2\tau_2(x_2 - \xi_2) & m_2 - b_2\tau_2(x_2 - \xi_2) & - c_2\tau_2(x_2 - \xi_2) & \cdots \\ - a_3\tau_3(x_3 - \xi_3) & - b_3\tau_3(x_3 - \xi_3) & m_3 - c_3\tau_3(x_3 - \xi_3) & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}^{-1}$$

where $\xi_1, \xi_2, \ldots$ *are the values of* $\mathrm{x}_1, \mathrm{x}_2, \ldots$ *which satisfy the equations* $\mathrm{U}_1 = \omega_1$, $\mathrm{U}_2 = \omega_2$, ...

In the course of the investigation a curious type of duality is brought to light, 'power 1' interchanging with 'power -1' and at the same time 'permanent' with 'determinant'. The simplest pair of the theorems is:

$$\begin{cases} \text{the coefficient of } x_1^{-1}x_2^{-1}x_3^{-1}\ldots \text{ in } U^{-1}U^{-1}U^{-1}\ldots \text{ is } |a_1b_2c_3\ldots|^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \qquad\quad + \\ \text{the coefficient of } x_1x_2x_3 \ldots \text{ in } U_1U_2U_3 \ldots \text{ is } |a_1b_2c_3\ldots|, \end{cases}$$

and the final pair consists of the widened generalization just quoted and the one got from it by dropping everywhere in it the exponent -1 and changing the determinant into the corresponding permanent.

EMINE, M. (1902/$_{11}$): HARDY, G. H. (1904/$_2$)

[Question 1317. *L'Intermédiaire des Math.*, v. p. 171: ix. pp. 297–298.]

[Question 15498. *Educ. Times*, lvii. p. 91: lx. p. 36.]

These concern old results, namely, Desnanot's of 1819 regarding vanishing aggregates of products of pairs of determinants and Cauchy's of 1812 regarding the adjugate of the product of two determinants (*Hist.*, i. pp. 121, 139–146).

NICOLETTI, O. (1902^{25}/$_{11}$)

[Alcuni teoremi sui determinanti. *Annali di Mat.*, (3) viii. pp. 287–297.]

The leading theorem here (pp. 287–291) expresses a determinant as an aggregate of four-factor products, the factors in each case being minors of four fixed arrays into which the determinant is partitioned by a horizontal and a vertical line, factors of order 0 with value 1 being not debarred. The actual expression obtained by Nicoletti is neither concise nor attractive, even in the case where two of the arrays are coaxial and complementary. The theorem is established naturally enough by a double application of Laplace's expansion-theorem. It is spoken of as an extension of Hesse's theorem of 1868 (*Hist.*, iii. pp. 28–29), and it clearly includes Arnaldi's first theorem of 1896 (*Hist.*, iv. p. 432). As a help towards obtaining the expansion in any particular case attention is drawn to the advantage gained by taking the four-factor products in a sequence dependent on the minors of one of the sub-arrays, say the first. For example, if the determinant be $|a_1b_2c_3d_4e_5|$ and the first array be 2-by-2, we should begin with the single product $|a_1b_2| \cdot |c_3d_4e_5| \cdot 1 \cdot 1$ whose minors are of the orders 2, 3, 0, 0, then take those in which the orders are 1, 2, 1, 1 and finally those in which the orders are 0, 1, 2, 2. Following on this practical hint come a few special deductions from the theorem: and then the last three pages are notably occupied in the reverse of specialization, the theorem being extended so as to hold for a determinant in which the number of sub-arrays is $h \times k$, the partitionment being made as before by horizontal and vertical lines.

MEYER, W. F. (1902/11)

[Aufgaben und Lehrsätze. Nr. 77, Nr. 78. *Archiv d. Math. u. Physik*, (3) iv. pp. 351–352.]

The first theorem here intended to be stated would seem to be: *If each element of the first* m-*line minor of a* 2m-*line determinant be multiplied by its cofactor in the latter, and the* m² *products thus obtained be summed, the result is the same as would have been got if instead of the first* m-*line minor we had taken the last.* In the original ' Unterdeterminante ' is used for ' signed complementary minor '. No proof is given, either of it or of a doubtful generalization.

The enunciation of the other theorem is, unfortunately, still less clear.

JOACHIMESCU, A. G. (1902/12)

[Inmulţirea determinanţilor. *Gazeta mat.* (Bucuresti), viii. pp. 77–80.]

A simply written demonstration of the multiplication-theorem, reversing the usual verification process—that is to say, taking

$$|a_1b_2|\cdot|\alpha_1\beta_2|,\quad |a_1b_2c_3|\cdot|\alpha_1\beta_2|,\quad |a_1b_2c_3|\cdot|\alpha_1\beta_2\gamma_3|$$

in order, and bringing out of them the product forms.

AURIC, A. (1902)

[Sur une propriété très générale des déterminants. *Bull.... Soc. Math. de France*, xxx. pp. 177–179.]

The property in question concerns the quotient of two determinants, the particular case dealt with being that in which the said determinants are cofactors of elements belonging to the last row of $|a_{11}\,a_{22}\,\ldots\,a_{nn}|$,—that is to say, the quotient of $A_{n,\,n-h}$ by A_{nn}. The subject is of old standing, the latest rediscovery regarding it having been made by H. v. Koch in 1890 (*Hist.*, iv. p. 419). The author we have now reached makes the degree of complication of the expression for the quotient depend on h.

MUIR, T. ($1903^{28}/_{2}$)

[Historical note in regard to determinants. *Nature*, lxvii. p. 512.]

Two matters are here dealt with: (1) the important part borne by Schweins (*Hist.*, ii. pp. 311–322) in the discovery of the theorem regarding the multiplication of an alternant by a symmetric function of the variables, (2) the unfair treatment of Sylvester in connection with the discovery of continuants, the offenders being S. Günther, E. Pascal, and the usually accurate E. Netto.

MUIR, T. ($1903^{28}/_{4}$)

[A general theorem giving expressions for certain powers of a determinant. *Report . . . S. African Association . . . Sci.*, i. pp. 229–232.]

The object here is to generalize a theorem—given in E. Pascal's textbook (1897), and there incorrectly called Hunyady's —for expressing the $(n+1)^{\text{th}}$ power of an n-line determinant in the form of a determinant of order $\frac{1}{2}n(n+1)$. The generalization is effected by viewing the determinant concerned as an eliminant, the result reached being that *If a set of* n *homogeneous linear equations in* n *unknowns be given, the determinant of the set being* Δ, *and there be formed another set consisting of all equations of the* r^{th} *degree derivable from the equations of the given set by multiplication among themselves, the determinant of the latter set is equal to* Δ^{C}, *where* C *is the combinatory number* $\mathrm{C}_{\mathrm{n}+\mathrm{r}-1,\,\mathrm{n}}$. Also, as a corrective to Pascal, the history of the case where $n = n$, $r = 2$ is carried back through Scholtz (1877) to Brill (1871), and the mode of treatment of these writers is explained.

In this connection it may be well to recall that Schläfli's preeminent claim to be viewed as the author of the general theorem was not discovered by Muir for a year or two after this and was only made known by him in 1911 (*Hist.*, ii. pp. 52–53). We may add further that the author might suitably have taken the opportunity to refer to the analogous case where $n = 2$, $r = r$ which E. B. Elliott, being like Pascal unacquainted with Schläfli's general theorem of 1851, had similarly in his textbook of 1895 assigned to Bruno (*Théorie des Formes Binaires*, pp. 228–229).

MUIR, T. ($1903^{28}/_{11}$, $1904/_{12}$)

[A third list of writings on determinants. *S. African Assoc. . . . Adv. of Sci.*, i. (Capetown), pp. 154–228: or *Quart. Journ. of Math.*, xxxvi. pp. 171–267.]

This list contains in all 895 titles, 194 belonging to the periods of the two preceding lists (*Hist.*, iv. pp. 12, 35) and 701 to the fifteen-year period 1886–1900.* There is a four-page introduction on the subject of such lists, the need for greater care and uniformity in the dating of scientific writings, the treatment of titles in the less common languages, and the need for the issue at intervals of a leaflet informing mathematicians of the libraries containing the serials referred to in such lists.

When at the request of the Editor the list was reprinted in the *Quart. Journ. of Math.* opportunity was taken to revise it and to append a list of corrections to the two preceding lists.

STEINITZ, E. ($1903^{29}/_{5}$)

[Ueber die linearen Transformationen welche eine Determinante in sich überführen. *Sitzungsb. d. Berliner Math. Ges.*, ii. pp. 47–52.]

The title implies of course that the author had set himself the problem of finding, for example, nine linear functions of the elements of $|u_1 v_2 w_3|$ which when substituted for the elements of $|x_1 y_2 z_3|$ would produce $|u_1 v_2 w_3|$ or a non-variable multiple of it, and that having succeeded in solving the problem he was about to give an account of his work and to state the result reached. Unfortunately after finishing his account he discovered that the said result was not new, having been formulated in the following manner by S. Kantor † quite six years before: *If each element of a determinant* $|A_{\alpha\beta}|$ *be replaced by a linear function of all the elements of the determinant, then and only then will* $|A_{\alpha\beta}|$

* The original intention had been to issue a list every five years; but this was thwarted by the appointment of the Compiler to the post of Superintendent-General of Education in Cape Colony, where for a number of years all available energy was absorbed in administration.

† *Sitzungsb. . . . Akad. d. Wiss.* (München), 1897, p. 370.

remain unaltered in substance when the linear functions have the form

$$\Sigma h_{\delta\beta} k_{\alpha\gamma} A_{\alpha\beta}.$$

As, however, no proof of this was given by Kantor,* Steinitz' paper with its full and clear exposition loses nothing in value.

HENSEL, K. ($1903^3/_6$)

[Bemerkungen zur Determinantentheorie. *Crelle's Journ.*, cxxvi. pp. 73–82.]

Hensel's main result, which follows up one attributed to Weierstrass by Frobenius, is that *if a function of the elements of an* m-*by*-n *array be such that it is linear and homogeneous in the elements of each row and changes sign with the interchange of any two rows, then it vanishes identically save when* m *and* n *are equal, in which case* (Weierstrass') *it is a constant multiple of the determinant of the array.* The line which his argument takes leads him on to give a proof (§ 2, pp. 76–78) of the extended multiplication-theorem, a proof (pp. 78–81) of Laplace's expansion-theorem, and a proof (p. 82) that the type of function under consideration, and therefore the function known as a determinant, is irreducible (i.e. irresolvable).

If the array which is the subject of the above theorem be square, and the function have the further property of becoming 1 when the diagonal elements are each 1 and the non-diagonal are 0, then the function must manifestly be a determinant. There thus arises the possibility of framing an alternative de-

* He may have thought that a proof was not needed, for he records instead his opinion that "the theorem ought to find a place at the beginning of every text-book of determinants". Whether this be a correct judgment or not there can be no harm in further lightening the beginner's task by giving him the connecting non-variable factor for the case where n is 2, the equality then to be proved being

$$\begin{vmatrix} \begin{pmatrix} h_{11}k_{11}v_{11} + h_{11}k_{21}v_{12} \\ + h_{12}k_{11}v_{21} + h_{12}k_{21}v_{22} \end{pmatrix}, & \begin{pmatrix} h_{11}k_{12}v_{11} + h_{11}k_{22}v_{12} \\ + h_{12}k_{12}v_{21} + h_{12}k_{21}v_{22} \end{pmatrix} \\ \begin{pmatrix} h_{21}k_{11}v_{11} + h_{21}k_{21}v_{12} \\ + h_{22}k_{11}v_{21} + h_{22}k_{21}v_{22} \end{pmatrix}, & \begin{pmatrix} h_{21}k_{12}v_{11} + h_{21}k_{22}v_{12} \\ + h_{22}k_{12}v_{21} + h_{22}k_{22}v_{22} \end{pmatrix} \end{vmatrix}$$

$$= |h_{11}h_{22}|\ |k_{11}k_{22}| \cdot \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}.$$

finition of the latter function, and of calling the three properties in question the 'defining characteristics'. When this actually came about is a little uncertain. Probably first in 1903, when Hensel edited a volume of Kronecker's lectures and so led to the intervention of Frobenius below referred to.

LOEWY, A. ($1903^1/_7$)

[Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen. *Transac. American Math. Soc.*, v. pp. 61–80.]

The introductory portion of this (pp. 61–66) is worthy of passing notice as being a re-exposition, with historical footnotes, of the determinantal implications of Hurwitz' paper of 1894 (*Hist.*, iv. p. 57), from which we learned how certain already familiar determinants—Zehfuss' of 1855, Schläfli's of 1851 and compound determinants—arise as the determinants of substitutions constructed in a variety of ways from the material of simpler substitutions. That such instances of fresh origin may be put to good use we have already seen in dealing with the subject of Latent Roots.

MUIR, T. ($1903^{12}/_8$)

[The theory of general determinants in the historical order of its development up to 1846. *Proceed. R. Soc. Edinburgh*, xxv. pp. 61–91.]

One of the papers ultimately embodied in Vol. II. of our History.

MEYER, W. F. ($1903^{25}/_{10}$)

[Aufgaben und Lehrsätze, Nr. 93. *Archiv d. Math. u. Phys.*, (3) vi. p. 339: viii. pp. 82–83.]

The proposition set here for proof and generalization may be formulated thus: *If* $|A_{11}\, A_{22}\, A_{33}|$ *be the adjugate of* $|a_{11}\, a_{22}\, a_{33}|$, *and* $\Pi(A)$ *be the product of all the elements of the*

former and $\Pi(a)$ *the like product in the case of the latter, then*

$$\left|\frac{1}{A_{11}}\cdot\frac{1}{A_{22}}\cdot\frac{1}{A_{33}}\right|\cdot\Pi(A)=-\left|a_{11}a_{22}a_{33}\right|^2\cdot\left|\frac{1}{a_{11}}\cdot\frac{1}{a_{22}}\cdot\frac{1}{a_{33}}\right|\cdot\Pi(a).$$

After verifying it C. Weltzien states that he had already done so in a school-program of 1886 (*Hist.*, iv. p. 33). The request for a generalization was not satisfied.

WEIERSTRASS, K. (1903)

[Zur Determinantentheorie. *Gesammelte Werke*, iii. pp. 271–287.]

This is not a reprint of a previously published paper of Weierstrass' but a report of a lecture delivered by him in the winter session of 1886–1887.* The object of the lecture, we are told, was to lay down a fresh definition of a determinant, and to show that the definition might be used to establish certain of the more important theorems. At the outset the determinant is given its usual introduction as the common denominator of the values of the unknowns in a set of linear equations, a three-line example being used to illustrate. Observed properties of it are then noted, and taken on trial as the defining characteristics of the corresponding function of n^2 elements—that is to say, the function which (1) is linear and homogeneous in the elements of each row, (2) changes only in sign if two rows be interchanged, (3) has 1 for its value when each diagonal element is 1 and each non-diagonal element is 0. The result of the trial being found satisfactory, instances are given, as promised, of its use, the most important being the establishment of the multiplication-theorem.

KRONECKER, L. (1903)

[Vorlesungen über Determinantentheorie. Band I. xii + 390 pp. Leipzig.]

In the seventeenth of these lectures (pp. 291–306), which

* An editorial footnote tells us that the reporter was Paul Günther, and that the manuscript of the report was then in the possession of the Mathematical Society of Berlin University: also, that several years before the date mentioned Weierstrass had shown his students how to deduce the basic theorems of determinants from the said three characteristic properties.

belong to the period 1883–1891, we find ourselves face to face with matter closely analogous to that of Paul Günther's version of a lecture of Weierstrass', published earlier in the same year but also belonging to a period much earlier than the year of publication. The question of priority, however, is not troublesome: there can be little doubt that the original idea of the tripartite definition was Weierstrass': indeed, the editor of the volume now before us says so in effect in his Preface (p. vi, footnote). All the same it would be a serious mistake to neglect Kronecker's exposition, which has all the fullness and clearness that characterize the rest of the volume. Perhaps the student's best course would be to place the two accounts side by side, and study them together.

YOUNG, A. ($1903/_{12}$)

[The expansion of the n^{th} power of a determinant. *Messenger of Math.*, xxxiii. pp. 113–116.]

The title here is apt to raise undue hopes. What we are given is the result of a successful attempt to find an expression for the typical term of

$$(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1)^n.$$

The introduced letters of the expression are of course much conditioned, and there is not more symmetry than could fairly be expected. A solution is also given of the like problem for the corresponding permanent.

CARLINI, L. (1903, 1904)

[Nuove considerazioni sopra le permutazioni. *Periodico di Mat.*, xix. pp. 32–38.]

[Sopra i sistemi ordinati di permutazione. *Il Pitagora*, x. pp. 134–137.]

The feature of a permutation that is here kept prominently in view is the number of its inversions of order: for example, the number of things permuted being 4, the 24 permutations of them are classified as being

1 with no inversion
3 with 1 inversion
5 with 2 inversions
6 with 3 ,,
5 with 4 ,,
3 with 5 ,,
1 with 6 ,,

It is hardly correct of the author to describe his considerations as being new, for we have twice already seen the subject dealt with by others, namely by Bourget in 1871 (*Hist.*, iii. p. 33), and by Muir in 1899 (*Hist.*, iv. pp. 72–73). Of the fresh results now reached one of the more notable is a rule for the calculation of the number of the said permutations in any given case—that is to say, a companion to Muir's rule of 1899. In the author's words the new rule is: When the number of elements permuted is n and the number of inversions is r, the number sought is equal to the number of positive integral solutions of the set of equations

$$x_1 + x_2 + \ldots + x_{n-1} = r, \text{ where } x_h \leq h;$$

and its companion * is that the number sought is the coefficient of x^r in the expansion of

$$(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)\big/(1-x)^n.$$

HADAMARD, J. (1903)

[Leçons sur la propagation des ondes et les équations de l'hydrodynamique. 375 pp. Paris.]

On pp. 13–14 of this book of lectures the author incidentally establishes anew Desplanques' theorem of 1886 (*Hist.*, iv. p.37) on the non-evanescence of a determinant, namely, *If in every row of a determinant the modulus of the diagonal element is greater than the sum of the moduli of the other elements, the determinant cannot vanish.*

For completeness' sake we ought to have referred above to a proof given by Minkowski † of Lévy's less general theorem of 1881 (*Hist.*, iv. p. 457).

* *Proceed. R. Soc. Edinburgh*, xxii. p. 456.
† Göttinger *Nachrichten*, 1900, pp. 90–91.

NESBITT, A. M. ($1904^1/_1$)

[Question 15487. *Educ. Times*, lvii. pp. 41, 239: or *Math. from Educ. Times*. (2) vi. pp. 86–87.]

The equality given here is

$$\begin{vmatrix} a & . & . & l & . & . \\ . & b & . & . & m & . \\ . & . & c & . & . & n \\ g & . & . & d & . & . \\ . & h & . & . & e & . \\ . & . & k & . & . & f \end{vmatrix} = (ad - gl)(be - hm)(cf - kn):$$

it may be viewed as expressing the product of any three determinants of the second order in a specially interesting form of six-line determinant.

HORDYŃSKI, L. ($1904/_3$)

[On partially transformed determinants (In Polish). *Wiadomosci mat.*, viii. pp. 177–190.]

This paper covers practically the same ground as the first half of Hunyady's paper of 1880 (*Hist.*, iv. pp. 204–205).

BAKER, H. F. ($1904/_4$)

[Note on Sylvester's theorems on determinants. *Collected Math. Papers* of J. J. Sylvester, i. pp. 647–650.]

This is a re-exposition, based on the theory of Cayleyan matrices, of a series of theorems that are mainly Sylvester's of March and August, 1851 (*Hist.*, ii. pp. 58–62, 193–197). It opens with the definition of the product of an m-by-n array and an n-by-r array: shows that a minor of such a product is itself the product of two arrays: and passes on to the multiplication of two and of three determinants. As an illustration of this last the linear transformation of a quadric is taken, and a re-examination is made of Sylvester's theorem regarding a minor of the discriminant of the quadric resulting from the transformation. Then the idea of mutual inverseness is introduced (*Hist.*, ii.

pp. 85–87: iv. pp. 13–14), and is given an important rôle in the evolution of a series of theorems—Cauchy's regarding the product of the r^{th} and $(n - r)^{\text{th}}$ compounds, Sylvester's expressing the r^{th} compound of A as a power of A, and Jacobi's regarding a minor of the adjugate. Of course the terms 'adjugate' and 'compound' are not mentioned in the presence of 'inverse', the apotheosis of which is reached when Sylvester's theorem expressing the product of two determinants as a sum of like products is handed to us in the guise of a statement that two matrices

$$\left(\frac{|a, b|}{A}\right), \quad \left(\frac{|b, a|}{B}\right)$$

are inverse.

The proof of the remaining theorem is new not only in form, and is in substance reproduced by us in the chapter on Compounds.

MUIR, T. ($1904^{20}/_{4}$)

[The three-line determinants of a three-by-six array. *Proceed. R. Soc. Edinburgh*, xxv. pp. 364–371.]

The subject here is a minute branch of what we have called 'vanishing aggregates of products of pairs of determinants'. The array being

$$\begin{matrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \end{matrix}$$

the notation purposely devised for the determinants is

$$\left.\begin{matrix} & |a_1b_2c_3| & \\ |a_1b_2f_3| & |b_1c_2g_3| & |c_1a_2h_3| \\ |a_1b_2g_3| & |b_1c_2h_3| & |c_1a_2f_3| \\ |a_1b_2h_3| & |b_1c_2f_3| & |c_1a_2g_3| \end{matrix}\right\} \equiv \left\{\begin{matrix} & 0 & \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}\right.$$

$$\left.\begin{matrix} & |f_1g_2h_3| & \\ |c_1g_2h_3| & |a_1h_2f_3| & |b_1f_2g_3| \\ |c_1h_2f_3| & |a_1f_2g_3| & |b_1g_2h_3| \\ |c_1f_2g_3| & |a_1g_2h_3| & |b_1h_2f_3| \end{matrix}\right\} \equiv \left\{\begin{matrix} & 0' & \\ 1' & 2' & 3' \\ 4' & 5' & 6' \\ 7' & 8' & 9'; \end{matrix}\right.$$

and the first theorem is to the effect that the product of two complementary determinants is expressible in six different ways as an aggregate of three similar products, there being however in all only nine different products involved; for example, the two triads of equivalents for 00′ are

$$\begin{array}{ll} 88' + 22' + 55', & 88' + 66' + 11' \\ 66' + 99' + 33', & 22' + 99' + 44' \\ 11' + 44' + 77', & 55' + 33' + 77', \end{array}$$

the second triad being got from the first by a row-column interchange. These relations are next viewed as equivalences of binomials,

$$\text{e.g.}\quad 00' - 77' = 11' + 44' = 33' + 55',$$

and a different classification arrived at, the binomials being also incidentally given a new form,

$$\text{e.g.}\quad \begin{vmatrix} |a_1b_2| & |a_2b_3| & |a_3b_1| \\ |f_1g_2| & |f_2g_3| & |f_3g_1| \\ |c_1h_2| & |c_2h_3| & |c_3h_1| \end{vmatrix} = 11' + 44'.$$

In similar fashion the products of non-complementary three-line minors are discussed, the result being that each of the nine such products can be expressed in one and only one way as a sum or difference of two other such products,

$$\text{e.g.}\quad 01' = 23 - 59.$$

In a concluding note on products of *three* factors the curious equalities

$$\begin{vmatrix} 1 & 8 & 6 \\ 4 & 2 & 9 \\ 7 & 5 & 3 \end{vmatrix} = 000', \qquad \begin{vmatrix} 1' & 8' & 6' \\ 4' & 2' & 9' \\ 7' & 5' & 3' \end{vmatrix} = 0'0'0$$

are drawn attention to, as also their connection with what precedes.

CARLINI, L. (1904/6)

[Due teoremi sui determinanti. *Giornale di Mat.*, xliii. pp.63–67.]

In reality there is only one theorem of note dealt with here, the second so-called being a mere corollary to the other. The

latter, too, dates back to 1879 (*Hist.*, iii. pp. 80, 82), its first appearance being in the guise of an extension of Hermite's theorem of 1849 (*Hist.*, ii. p. 46), and therefore being known under the designation of a 'condensation-theorem'. It now turns up in its old character; indeed, with Carlini the idea of condensation is emphasized. His enunciation of it is: *Save for a connecting factor, any* n*-line determinant may be expressed as an* (n — m + 1)*-line determinant whose elements are* m*-line minors, the said connecting factor being a product of* (m — 1)*-line minors.* For example, when n is 5 and m is 3,

$$|a_1b_2c_3d_4e_5| \cdot \{|a_2b_3| \cdot |a_3b_4|\} = \begin{vmatrix} |a_1b_2c_3| & |a_2b_3c_4| & |a_3b_4c_5| \\ |a_1b_2d_3| & |a_2b_3d_4| & |a_3b_4d_5| \\ |a_1b_2e_3| & |a_2b_3e_4| & |a_3b_4e_5| \end{vmatrix}.$$

As thus viewed the theorem receives here its fullest treatment, three preparatory pages being devoted to auxiliary results regarding combinations. Its other aspect must not, however, be overlooked, namely, as concerning a special minor of the m^{th} compound that is resolvable into $n - m$ factors (*Hist.*, iv. p. 22).

Another important fact about it is that in the strictly technical sense it is an 'extensional'. For, taking only the above instance, and writing it in the equivalent form

$$|a_3b_4c_5d_1e_2| \cdot \{|a_3b_2| \cdot |a_3b_4|\} = \begin{vmatrix} |a_3b_1c_2| & |a_3b_2c_4| & |a_3b_4c_5| \\ |a_3b_1d_2| & |a_3b_2d_4| & |a_3b_4d_5| \\ |a_3b_1e_2| & |a_3b_2e_4| & |a_3b_4e_5| \end{vmatrix}$$

we obtain, on deleting everywhere a_3,

$$|b_4c_5d_1e_2| \cdot b_2b_4 = \big|\, |b_1c_2| \quad |b_2d_4| \quad |b_4e_5| \,\big|,$$

which is simply Hermite's original equality.

BRAND, E. ($1904^{15}/_9$)

[Un symbole d'opération dans le calcul des dérivées. *L'Enseignement Math.*, vi. pp. 457–459.]

As an illustration of the use of his symbol the author returns to determinants: but the result is essentially what he had reached in 1896 (*Hist.*, iv. p. 64).

RUSK, W. J. (1904)

[The n^{th} derivative of a determinant whose constituents are functions of a given variable. *American Math. Journ.*, xii. p. 85.]

The result here is essentially that of F. G. Teixeira's paper of 1880 and of Brand's paper of 1896 (*Hist.*, iv. pp. 5, 64). The proof given is gradational.

MUIR, T. ($1905^{12}/_{6}$)

[The theory of general determinants in the historical order of development up to 1852. *Proceed. R. Soc. Edinburgh*, xxv. pp. 908–947.]

One of the papers ultimately embodied in Vol. II of our History.

HERTWIG, A. ($1905^{13}/_{6}$)

[Beziehungen zwischen Symmetrie und Determinanten in einigen Aufgaben der Fachwerktheorie. *Festschrift . . . Adolph Wüllner* . . . pp. 194–213.]

The author, an architect apparently and certainly a student of the Statics of Building Construction, gives in an interesting way his experiences of determinants of sets of linear equations met with in the designing of framed structures; and what attracts him is the fact that the introduction of symmetry into a structure is accompanied by something analogous in the corresponding determinant. After a two-page introduction, mainly regarding the sets of equations, he brings forward in order the relevant determinants. First comes the circulant, then Puchta's block circulant of order 2^3 (*Hist.*, iii. p. 388), and the determinant got by circulating the arrays of two 3-line circulants (*Hist.*, iv. p. 385), the immediate object aimed at in every case being resolution into factors. Following on these comes something fresher, the factorization of the determinant

$$\begin{vmatrix} a_1 & b_1 & a_3 & b_5 & a_5 & b_3 & a_7 & b_7 \\ a_2 & b_2 & a_6 & b_4 & a_4 & b_6 & a_8 & b_8 \\ a_3 & b_3 & a_1 & b_7 & a_7 & b_1 & a_5 & b_5 \\ a_4 & b_4 & a_8 & b_2 & a_2 & b_8 & a_6 & b_6 \\ a_5 & b_5 & a_7 & b_1 & a_1 & b_7 & a_3 & b_3 \\ a_6 & b_6 & a_2 & b_8 & a_8 & b_2 & a_4 & b_4 \\ a_7 & b_7 & a_5 & b_3 & a_3 & b_5 & a_1 & b_1 \\ a_8 & b_8 & a_4 & b_6 & a_6 & b_4 & a_2 & b_2 \end{vmatrix}$$

whose odd-numbered columns are each composed of the elements $a_1, a_2, \ldots, a_8$ and whose other columns are each composed of the elements $b_1, b_2, \ldots, b_8$. To this he devotes considerable space, and succeeds in resolving it into four determinants of the second order. For ourselves we note that it is readily transformable into

$$\begin{vmatrix} a_1 & b_1 & a_3 & b_3 & a_5 & b_5 & a_7 & b_7 \\ a_2 & b_2 & a_6 & b_6 & a_4 & b_4 & a_8 & b_8 \\ a_3 & b_3 & a_1 & b_1 & a_7 & b_7 & a_5 & b_5 \\ a_6 & b_6 & a_2 & b_2 & a_8 & b_8 & a_4 & b_4 \\ a_5 & b_5 & a_7 & b_7 & a_1 & b_1 & a_3 & b_3 \\ a_4 & b_4 & a_8 & b_8 & a_2 & b_2 & a_6 & b_6 \\ a_7 & b_7 & a_5 & b_5 & a_3 & b_3 & a_1 & b_1 \\ a_8 & b_8 & a_4 & b_4 & a_6 & b_6 & a_2 & b_2 \end{vmatrix},$$

that is to say, into a block circulant of two four-line arrays each of which is itself a circulant of two two-line arrays: or, what is more readily helpful, into

$$\begin{vmatrix} a_1 & b_1 & a_3 & b_3 & b_5 & a_5 & b_7 & a_7 \\ a_2 & b_2 & a_6 & b_6 & b_4 & a_4 & b_8 & a_8 \\ a_3 & b_3 & a_1 & b_1 & b_7 & a_7 & b_5 & a_5 \\ a_4 & b_4 & a_8 & b_8 & b_2 & a_2 & b_6 & a_6 \\ a_6 & b_6 & a_2 & b_2 & b_8 & a_8 & b_4 & a_4 \\ a_5 & b_5 & a_7 & b_7 & b_1 & a_1 & b_3 & a_3 \\ a_8 & b_8 & a_4 & b_4 & b_6 & a_6 & b_2 & a_2 \\ a_7 & b_7 & a_5 & b_5 & b_3 & a_3 & b_1 & a_1 \end{vmatrix},$$

which is centrosymmetric and resolvable into

$$\begin{vmatrix} a_1+a_7 & b_1+b_7 & a_3+a_5 & b_3+b_5 \\ a_2+a_8 & b_2+b_8 & a_4+a_6 & b_4+b_6 \\ a_3+a_5 & b_3+b_5 & a_1+a_7 & b_1+b_7 \\ a_4+a_6 & b_4+b_6 & a_2+a_8 & b_2+b_8 \end{vmatrix} \cdot \begin{vmatrix} a_1-a_7 & b_1-b_7 & a_3-a_5 & b_3-b_5 \\ a_2-a_8 & b_2-b_8 & a_6-a_4 & b_6-b_4 \\ a_3-a_5 & b_3-b_5 & a_1-a_7 & b_1-b_7 \\ a_4-a_6 & b_4-b_6 & a_8-a_2 & b_8-b_2 \end{vmatrix}$$

that is to say, into a pair of determinants each of which is centrosymmetric and therefore further resolvable. The special example

$$\begin{vmatrix} a & b & \cdot & \cdot & -e & -f & \cdot & \cdot \\ -b & -a & \cdot & \cdot & f & -e & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot & -e & -f & \cdot \\ \cdot & -b & -a & \cdot & \cdot & f & -e & \cdot \\ \cdot & \cdot & a & b & \cdot & \cdot & -e & -f \\ \cdot & \cdot & -b & -a & \cdot & \cdot & f & -e \\ b & \cdot & \cdot & a & -f & \cdot & \cdot & -e \\ -a & \cdot & \cdot & -b & -e & \cdot & \cdot & f \end{vmatrix} = -(a^2-b^2)\,(4e^2)\,(4a^2)\,(f^2+e^2)$$

is added: and then the remainder of the paper (pp. 205–213) is occupied with the consideration and diagrammatic illustration of the framed structures in which the sets of equations and their determinants originated.

FROBENIUS, G. (1905/7)

[Zur Theorie der linearen Gleichungen. *Crelle's Journ.*, cxxix. pp. 175–180.]

As an appendix to this paper but quite unconnected with it there is a note (pp. 179–180) on the three defining characteristic properties of a determinant. Frobenius says that so far back as 1864 Weierstrass had used the definition in his lectures, and that in the summer of 1874 he himself (Frobenius) in publicly discussing the subject had as a consequence already proved the first two of the three theorems dealt with by Hensel in his paper of 1903 (see above).

MUIR, T. ($1905^{19}/_{7}$)

[The Theory of Determinants in the Historical Order of Development. Part I (second edition) General Determinants up to 1841: Part II, Special Determinants up to 1841. xii + 492 pp. London.]

The first complete volume of our History.

SOMMERVILLE, D. M. Y. ($1905^{10}/_{11}$)

[On the number of independent conditions involved in the vanishing of a rectangular array. *Proceed. Edinburgh Math. Soc.*, xxiv. pp. 2–6.]

This is a fresh and interesting investigation written without acquaintance with previous papers on the subject, in particular, Sylvester's of 1850 in regard to any p-by-q array and to a square axisymmetric array (*Hist.*, ii. pp. 50–52: iv. p. 67). A new result is added, namely, that in a p-by-q array of which the first q-by-q determinant is axisymmetric, the number of conditions in question—that is to say, for the co-evanescence of k-line minors—is $\frac{1}{2}(q - k + 1)(2p - q - k + 2)$. At the close all the results are cunningly summarized in one enunciation, namely: *If* $\phi(\mathrm{p}, \mathrm{q})$ *be the number of different elements in either of the two kinds of* p*-by-*q *arrays considered, then the number of* k*-line determinants in the arrays is* $\phi(\mathrm{p}_k, \mathrm{q}_k)$, *and the number of conditions in question is*

$$\phi(p - k + 1, q - k + 1).$$

In illustration of the author's mode of proof it will be helpful to use again the simple example dealt with by Spottiswoode in 1853 (*Hist.*, ii. pp. 83–84), namely, the 4-by-7 array:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 \end{matrix}$$

in which it is given that $|a_1b_2c_3d_4| = |a_1b_2c_3d_5| = |a_1b_2c_3d_6| = |a_1b_2c_3d_7| = 0$. By performing the operation

$$\mathrm{row}_1 \cdot |b_1c_2d_3| - \mathrm{row}_2 \cdot |a_1c_2d_3| + \mathrm{row}_3 \cdot |a_1b_2d_3| - \mathrm{row}_4 \cdot |a_1b_2c_3|$$

there results on addition the row

$$|a_1b_1c_2d_3|,\ |a_2b_1c_2d_3|,\ |a_3b_1c_2d_3|,\ |a_4b_1c_2d_3|,\ |a_5b_1c_2d_3|,\ |a_6b_1c_2d_3|,\ |a_7b_1c_2d_3|,$$

which is

$$0, \qquad 0, \qquad 0, \qquad 0, \qquad 0, \qquad 0, \qquad 0,$$

because each of the first three has two columns identical, and the remaining four are given so. The array is thus proved to be evanescent unless all the multipliers are 0,

$$\text{i.e. unless} \left\| \begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix} \right\| = 0.$$

MUIRHEAD, R. F. (1905/11)

[Proof of the multiplication-theorem for determinants. *Messenger of Math.*, xxxv. pp. 151–152.]

The procedure here is not without freshness: in type it somewhat resembles Le Paige's of 1881 (*Hist.*, iv. p. 5).

BAKER, R. P. (1906/1)

[On the identical relations between the determinants of an array. *American Math. Monthly*, xiii. pp. 1–10, 30–33.]

This is an article of a type that might with advantage be more common. The author selects a subject which has grown merely by unplanned accretions—the subject of 'vanishing aggregates of determinant products'—and makes a study of it with the object of presenting it as an organized whole. After a glance at the history of the early isolated cases, and a redetermination of the number of independent relations between the m-line minors of an m-by-n array (*Hist.*, iii. p. 68) the main business is entered on. The unifying wand, so to speak, is Cayley's procedure of 1877 (*Hist.*, iii. pp. 65–66) where the aggregate of products concerned appears as a Laplace development of a specially constructed vanishing determinant. The different types of aggregates are taken up in succession, the number of terms

in each being given, the characteristics of the single equivalent determinant set forth, and the number of the aggregates belonging to each type calculated.

With additional pains spent on the exposition the outcome would have been of still greater interest.

PETR, K. (1906)

[Nékolik poznámek o determinantech. *Časopis pro péstování math. a. fys.*, xxxv. pp. 311–321: or, in Magyar, *Math. és Phys. Lapok*, xv. pp. 353–365: or, in German, *Math.-naturw. Berichte aus Ungarn*, xxv. pp. 95–105.]

The first section of this shows how to arrive at relations between m-line minors of a vanishing $2m$-line determinant whose non-zero rank is m. The method employed will be fully understood from considering the case where m is 2 and the vanishing determinant is $|a_1b_2c_3d_4|$, or D say. Constructing from D the determinant

$$\begin{vmatrix} a_1 & a_2 & \lambda a_3 & \mu a_4 \\ b_1 & b_2 & \lambda b_3 & \mu b_4 \\ c_1 & c_2 & xc_3 & xc_4 \\ d_1 & d_2 & xd_3 & xd_4 \end{vmatrix}, \quad \text{or } \Delta \text{ say,}$$

we see that the latter must have $x - \lambda$ for a factor: for on putting in it $x = \lambda$ the factor λ is removable and its cofactor is a determinant with a 4-by-3 vanishing array. Similarly, of course, we see that $x - \mu$ is a factor: and therefore

$$\Delta = M \cdot (x - \lambda)(x - \mu)$$

where M is independent of x, λ, μ. This shows that $x^2, -\lambda x, -\mu x$ $\lambda\mu$ have in Δ all the same cofactor, and consequently that

$$\begin{aligned} &|a_1b_2|\,|c_3d_4| = |a_1b_3|\,|c_2d_4| - |a_2b_3|\,|c_1d_4| \\ = -&|a_1b_4|\,|c_2d_3| + |a_2b_4|\,|c_1d_3| = |a_3b_4|\,|c_1d_2|, \end{aligned}$$

as desired.

Attention might have been drawn to the relation existing between these equalities and the single equality which holds when the nullity of $|a_1b_2c_3d_4|$ is only 1, namely,

$$|a_1b_2|\,|c_3d_4| - |a_1b_3|\,|c_2d_4| + |a_2b_3|\,|c_1d_4| + |a_1b_4|\,|c_2d_3| - |a_2b_4|\,|c_1d_3| + |a_3b_4|\,|c_1d_2| = 0.$$

In the third section he re-establishes Zehfuss' composition-theorem of 1858, properly objecting to attribute it to Kronecker, but saying nothing of Zehfuss: also Siacci's theorem of 1872 regarding the resolution of

$$|(\lambda a_{11} + \mu b_{11}) \ldots (\lambda a_{nn} + \mu b_{nn})|.$$

NICOLETTI, O. (1906 $^{13}/_5$)

[Su un teorema di Kronecker della teoria dei determinanti. *Rendic. del Circolo Mat.* (Palermo) . . ., xxii. pp. 112–116.]

The theorem in question is that of 1869 (*Hist.*, iii. p. 191), but now made to assume the guise of a test for an m-by-n array being of non-zero rank r, the necessary and sufficient condition being stated to be that *the array have a non-zero* r-*line minor whose* (r + 1)-*line majors all vanish.* What the writer contributes is a proof, founded like Kronecker's second proof, on the theory of systems of moduli of rational integral functions. He leads up also to a second proof of his own theorem of 1902.

KÜRSCHÁK, J. (1906 $^{15}/_5$)

[Sur l'irréductibilité de certains déterminants. *L'Enseignement Math.*, viii. pp. 207–208: or, in Magyar, *Math. és Phys. Lapok*, xv. pp. 1–2.]

The determinants in question are the general n-line determinant whose elements are independent indeterminates and the n-line axisymmetric determinant with elements of the same type. The base of Kürschák's reductio-ad-absurdum argument is that the substitution of 0 or 1 for any number of the elements in either determinant will not do away with its supposed resolvability. It thus suffices to consider only the determinant whose main diagonal is left untouched, whose adjacent minor diagonals are composed of units, and whose other elements are zeros: and then a property of simple continuants comes to his aid (*Hist.*, iii. p. 412).

With this may profitably be compared a proof like Böcher's of the following year.*

PASCAL, E. (1906/$_7$)

[Sui determinanti composti e su di un covariante estensione dell' Hessiano di una forma algebrica. *Rendic. del Circolo Mat.* (Palermo), xxii. pp. 371–382.]

The generalization here made on the Schläfli or so-called 'Scholtz-Hunyady' determinant is Muir's rediscovery of 1903. The proof given, however, is quite fresh, being dependent on the theory of quantics; and it carries with it the interesting fact that the said determinant is "the discriminant of the k^{th} power of a bilinear quadric". To be more precise, there exists an equality which, when k is 2 and the basic determinant is $|\,a_1 b_2\,|$, is best put in the form

$$\left\{\begin{array}{cc|c} x & y & \\ \hline a_1 & a_2 & \xi \\ b_1 & b_2 & \eta \end{array}\right\}^2 = \begin{array}{ccc|l} x^2 & xy & y^2 & \\ \hline a_1^2 & 2a_1a_2 & a_2^2 & \xi^2, \\ 2a_1b_1 & 2(a_1b_2 + a_2b_1) & 2a_2b_2 & \xi\eta \\ b_1^2 & 2b_1b_2 & b_2^2 & \eta^2 \end{array}$$

where, be it observed, the determinant of the 3-line array on the right is equal, not to $|\,a_1b_2\,|^3$, but to $2|\,a_1b_2\,|^3$. Our new acquisition thus is the ability to express the k^{th} power of a bilinear as a bilinear: and the determinant of the square array of the latter can be made exactly a Schläfli determinant by merely removing a factor or factors to the outside: for example, when k is 3, the right-hand expression is

$$\begin{array}{cccc|l} x^3 & x^2y & xy^2 & y^3 & \\ \hline a_1^3 & 3a_1^2a_2 & 3a_1a_2^2 & a_2^3 & \xi^3 \\ a_1^2b_1 & a_1^2b_2 + 2a_1a_2b_1 & a_2^2b_1 + 2a_1a_2b_2 & a_2^2b_2 & 3\xi^2\eta \\ a_1b_1^2 & a_2^2b_1 + 2a_1b_1b_2 & a_1b_2^2 + 2a_2b_1b_2 & a_2b_2^2 & 3\xi\eta^2 \\ b_1^3 & 3b_1^2b_2 & 3b_1b_2^2 & b_2^3 & \eta^3 \end{array}$$

(see *Hist.*, ii. pp. 52–53).

* *Introduction to Higher Algebra*, pp. 176–177.

MUIR, T. ($1906^3/_9$)

[The sum of the r-line minors of the square of a determinant. *Proceed. R. Soc. Edinburgh*, xxvi. pp. 533–539.]

The theorem established on this subject is: *The sum of the* r-*line minors of the square of any* n-*line determinant* Δ *is equal to the sum of* $(\mathrm{n})_r$ *squares, each of which is the square of the sum of the* r-*line minors formable from a set of* r *columns of* Δ. Attention is drawn to the special cases for which r is 1 and r is $n-1$, an instructive corroborating proof of the latter being, for the 3rd order (*Hist.*, iv. p. 430)

$$-\begin{vmatrix} \cdot & 1 \quad 1 \quad 1 \\ 1 & \\ 1 & |a_1b_2c_3|^2 \\ 1 & \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}^2 + \begin{vmatrix} 1 & a_1 & a_3 \\ 1 & b_1 & b_3 \\ 1 & c_1 & c_3 \end{vmatrix}^2 + \begin{vmatrix} 1 & a_2 & a_3 \\ 1 & b_2 & b_3 \\ 1 & c_2 & c_3 \end{vmatrix}^2$$

$$= (A_3 + B_3 + C_3)^2 + (A_2 + B_2 + C_2)^2 + (A_1 + B_1 + C_1)^2.$$

Passage is then made from Δ^2 to $\Delta_1\,\Delta_2$, the more general theorem being: *The sum of the* r-*line minors of* $\Delta_1\,\Delta_2$ *is the sum of all possible products having for their first factor an* r-*line minor of* Δ_1 *and for their second the corresponding minor of* Δ_2. There are also given, though rather unnecessarily (*Hist.*, ii. p. 120) the two corresponding theorems * for the sums of the r-line *coaxial* minors.

MUIR, T. ($1906^{19}/_9$)

[A fourth list of writings on determinants. *Quarterly Journ. of Math.*, xxxvii. pp. 237–264.]

This list contains in all 313 titles, 97 belonging to the periods of the three preceding lists, and 216 to the five-year period 1901–1906.

* One of these is attributed mistakenly to Picquet by E. Pascal in § 15 of his text-book.

KÜRSCHÁK, J. (1906/11)

(See under this heading in Chap. XI.)

MUIR, T. (1906 12/11)

[The minors of a product determinant. *Proceed. R. Soc. Edinburgh*, xxvii. pp. 79–87.]

What is really dealt with here is the *condensation* of the said minors. Taking in the first place the primary minors, the author establishes the fact that *the minors of highest order in a product determinant are condensable into the same form as the minors of the lowest order* (i.e. *the elements*). By calling in the aid of the ordinary notation for an adjugate this can be put more definitely as follows: *The cofactor of any element in the determinant which equals the product*

$$|a_{11}\dots a_{nn}|\cdot|b_{11}\dots b_{nn}|\cdot|c_{11}\dots c_{nn}|\dots$$

is got from the expression for the said element by changing every letter in it into the corresponding capital letter. For example, in the determinant equal to $|a_1b_2c_3|\cdot|f_1g_2h_3|$ the $(1, 1)^{\text{th}}$ element being $(a_1, a_2, a_3)(f_1, g_1, h_1)$, the cofactor of this element is $(A_1, A_2, A_3)(F_1, G_1, H_1)$: and in the determinant equal to

$$|a_1b_2c_3|\cdot|f_1g_2h_3|\cdot|m_1n_2r_3|$$

the $(2, 3)^{\text{th}}$ element being

$$\begin{array}{ccc|c} b_1 & b_2 & b_3 & \\ \hline f_1 & g_1 & h_1 & m_3 \\ f_2 & g_2 & h_2 & n_3 \\ f_3 & g_3 & h_3 & r_3 \end{array}, \quad \text{its cofactor is} \quad \begin{array}{ccc|c} B_1 & B_2 & B_3 & \\ \hline F_1 & G_1 & H_1 & M_3 \\ F_2 & G_2 & H_2 & N_3 \\ F_3 & G_3 & H_3 & R_3 \end{array}$$

An evident corollary is that *the adjugate of the product of any number of determinants is not only equal to, but is identical in form with the product of their adjugates* (*Hist.*, i. p. 121: iii. p. 8).

In the next place minors other than those of the highest order are tackled, and, though at first sight the result obtained may seem different, in form it is not really so. It stands thus: *Every minor of the determinant equal to* $\Delta_1\Delta_2\dots\Delta_z$, *say the*

r-*line minor in rows* $\alpha, \beta, \gamma, \ldots$ *and in columns* $\alpha', \beta', \gamma', \ldots$ *is expressible as a bipartite whose elements are in order*

(1) *the* r-*line minors formable from* rows $\alpha, \beta, \gamma, \ldots$ *of* Δ_1,
(2) ,, ,, ,, *from* Δ'_2,
(3) ,, ,, ,, *from* Δ_3,
. . . .
(z) ,, ,, ,, *from columns* $\alpha', \beta', \gamma', \ldots$ *of* Δ_z.

For example, the two-line minor belonging to rows 2, 4 and columns 1, 2 of the determinant equal to

$$| a_1 b_2 c_3 d_4 | \cdot | f_1 g_2 h_3 k_4 | \cdot | m_1 n_2 r_3 s_4 | \cdot | x_1 y_2 z_3 w_4 |$$

is the bipartite briefly representable by

Two-line minors of

$$\begin{Vmatrix} b_1 & b_2 & b_3 & b_4 \\ d_1 & d_2 & d_3 & d_4 \end{Vmatrix}$$

Two-line minors of

$$\begin{vmatrix} f_1 & g_1 & h_1 & k_1 \\ f_2 & g_2 & h_2 & k_2 \\ f_3 & g_3 & h_3 & k_3 \\ f_4 & g_4 & h_4 & k_4 \end{vmatrix}$$

Two-line minors of

$$\begin{vmatrix} m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix}$$

Two-line minors of

$$\begin{Vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{Vmatrix}$$

PASCAL, E. (1907/2)

(See under this heading in Chap. VIII)

BOULAD, F. (1907, 1909, 1914): MEHMKE, R. (1913): KÖNIG, D. (1915)

[Sur la résolution graphique des équations linéaires. *Nouv. Annales de Math.*, (4) vii. pp. 1–7.]

[Un procédé de calcul graphique des déterminants. *Assoc. franç. pour l'avancem. des sci.*, (Lille), pp. 95–100.]

[Nouveaux procédés d'élimination des inconnues dans un système d'équations linéaires. *Assoc. franç. pour l'avancem. des sci.*, (Havre), 17 pp.]

[Graphische Berechnung von Determinanten beliebiger Ordnung. *Zeitschrift f. Math. u. Phys.*, lxii. pp. 209–218.]

[Vonalrendzerek és determinánsok. *Math. és termesz. értesitö* (Budapest) xxxiii. pp. 221–229.]

As might be expected from a perusal of their titles these papers do not help in any way towards the extension of the theory of determinants. In the first place they are seen to belong to the applicational side of the subject, and further examination brings out the fact that they are mainly occupied with cases of evaluation where the elements are purely arithmetical.

QUINTILI, P. (1907/8)

[Sopra uno speciale determinante. *Periodico di Mat.*, (3) v. pp. 117–119.]

Notwithstanding the title the theorem here carefully established is more suitably classified along with those concerning determinants in general. It may be enunciated thus: *If the determinant got from* $|a_{11}\ a_{22}\ \ldots\ a_{nn}|$ *by increasing each diagonal element by* x *be differentiated with respect to* x, *its* m^{th} *differential coefficient will be found equal to* m! *times the sum of its* (n — m)-*line coaxial minors.* It is arrived at by comparing the expansion of the determinant according to powers of x with the like expansion as furnished by Taylor's theorem.

MUIR, T. (190815/6)

[The theory of general determinants in the historical order of development up to 1860. *Proceed. R. Soc. Edinburgh*, xxviii. pp. 676–702.]

One of the papers ultimately embodied in Vol. II of our History.

DIXON, A. L. ($1908^{14}/_{10}$, $1909^{8}/_{1}$)

[The eliminant of three quantics in two independent variables. *Proceed. London Math. Soc.*, (2) vii. pp. 49–69, 473–492.]

These two papers, because of their value in relation to the general subject of elimination, we have already placed in our list at the end of the chapter on Bigradients. Attention, however, must in addition be drawn to the wealth of eliminant forms presented in them, to the discussion of the expansion of Sylvester's unisignant (pp. 474–478) and especially to the extension given to Borchardt's mode of elimination. One of the said eliminants, that of three binary cubics, is probably the most widely-spread determinant hitherto printed, occupying the whole expanse of two facing pages (pp. 56–57).

BOUMAN, Z. P. (1908)

[Vraagstuk lxv. *Wiskundige Opgaven*, x. pp. 151–154.]

The subject here is the r^{th} adjugate of a determinant. It is clearly dealt with; but the results obtained do not add to Grusintzeff's work of 1891 (*Hist.*, iv. pp. 213–215).

MACLAGAN-WEDDERBURN, J. H. ($1908^{11}/_{12}$)

[On certain theorems in determinants. *Proceed. Edinburgh Math. Soc.*, xxvii. pp. 67–69.]

The determinants here briefly considered are those whose elements are hypercomplex numbers of the type

$$\xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_n e_n,$$

where the e's are linearly independent units: and the few theorems referred to are the analogues of fundamental theorems that hold when the elements are ordinary numbers. Deserving of special note is the paragraph dealing with the result of subjecting the elements to a linear transformation (cf. *Hist.*, iii. pp. 487–489).

KOWALEWSKI, G. (1909)

[Einführung in die Determinantentheorie. v + 550 pp. Leipzig.]

The preliminary exposition (pp. 6–18) leading up to the term-formation of a determinant is somewhat novel, the initial concept being that of a ' pairing ' of n things with n other things, the usual terms ' inversion ', ' interchange ', ' circular substitution ', &c., coming up for explanation in due course. The outcome is (p. 19) that

$$|\,a_{11}\; a_{22}\; .\; .\; .\; a_{nn}\,| \equiv \Sigma\,\mathrm{sgn}\mathfrak{p} \cdot p(\mathfrak{p})$$

where $p\mathfrak{p}$ is the product of the elements determined by the ' Paarung ' $\mathfrak{p}$, and sgn$\mathfrak{p}$ is the sign of the ' Paarung '.

Of more importance are two paragraphs (pp. 29–32) one concerning the properties which characterize a determinant as a function, and the other dealing with its continuity. Under the latter proof is given that *A vanishing determinant may be viewed as the limit of a succession of non-zero determinants*: and under the former a supposedly fresh characteristic is found prefixed to the hitherto recognized three, namely, that *A determinant is constructed from the elements by the operations of addition, subtraction and multiplication.*

The theorem regarding the multiplication of two m-by-n arrays receives full attention, a second proof, based on the representation of the product as a determinant of the $(m + n)^{\text{th}}$ order, being carefully worked out (*Hist.*, ii. p. 200).

TURNBULL, H. W. (1909$^5/_3$)

[The irreducible concomitants of two quadratics in n variables. *Transac. Cambridge Philos. Soc.*, xxi. pp. 197–240.]

In view of the nature of the subject one might well expect to have determinants utilized here, even freely. It is an agreeable surprise, however, to find wholly occupied with them a chapter of ten pages (pp. 201–210) under the heading " The symbolical notation and fundamental identities for n-ary forms ". The theorems signalized as ' fundamental identities '—thus indicating the important part played by determinants in the paper—are

practically all of the same type, belonging to that extensive sub-class whose subject is "aggregates of products of determinants" and whose origin dates back to Bezout and the year 1779 (*Hist.*, i. p. 51). The general importance of the theorems in question is further attested in a way which the author apparently was unaware of, namely, by the frequency with which they had been discussed by writers from the time of Bezout onward. Of this, perhaps, the best idea can be got for the early period by turning to the Index of Numbered Results which occupies p. 489 of the first volume of the History, and noting the references given under the numbers xxiii, xliv, xlix. On the other hand it cannot be denied that under this head even the leading textbooks had for many years been strikingly remiss.* As for the so-called 'symbolic notation' it may be roughly described as a modification and extension of that of Aronhold and Clebsch familiar to students of concomitants,† and it may be illustrated by saying that Sylvester's theorem of 1851 regarding a vanishing aggregate of products of pairs of determinants appears included in

$$\Sigma(a_n)(a_{n-1}\beta) = 0.$$

In addition to the beneficial fillip which the paper as a whole gives to the ordinary student of determinants, he will find it of value to note the way in which it proposes to pass on from products like Sylvester's just mentioned to products of three or more factors, a subject not touched on by any writer since Monge in 1809 (*Hist.*, i. p. 68). The value of R. P. Baker's paper of 1906 as a companion need hardly be recalled (*v.* above, pp. 42–43).

WELLSTEIN, J. ($1909^{17}/_{3}$)

[Kriterien für die Potenzen einer Determinante. *Math. Annalen*, lxvii. pp. 490–497.]

[Die Dekomposition der Matrizen. *Göttinger Nachrichten*, 1909, pp. 97–99.]

By the title of the first of these papers the author, we find, meant to indicate that the paper supplied a means for ascertain

* See *Proceed. R. Soc. Edinburgh*, xxxviii. p. 219 and *Hist.*, iv. p. 81.

† See, for example, Salmon's *Mod. Higher Alg.*, Lesson XIV.

ing whether any given integral function of the elements of a square array is or is not actually a power of the determinant of the array. The subject of the second is closely related thereto: indeed, the title of it is exactly the same as the heading of a section of the first. We thus at once learn that the author's field of investigation is not new; and, although he makes no reference to previous workers in the field, we have little difficulty in getting into touch with a paper of Mertens' of 1893 (see above), and thence to three others of the same writer. What, however, is of most interest to us is the fact that the two modes of investigation are not alike, Wellstein's being based on the so-called Kronecker definition of a determinant to which we had occasion to draw attention a few pages back under Hensel (1903).

MUIR, T. ($1909/_6$)

[The superadjugate determinant, and skew determinants having a univariant diagonal. *Proceed. R. Soc. Edinburgh*, xxix. pp. 668–686.]

Only the first three pages of this longish paper concern general determinants. They open with a definition, *the* SUPERADJUGATE *of any determinant* $|a_{11}\, a_{22} \ldots a_{nn}|$, *or* Δ *say, being defined as the determinant*

$$\begin{vmatrix} 2a_{11}A_{11}-\Delta & 2a_{22}A_{12} & 2a_{33}A_{13} & \ldots \\ 2a_{11}A_{21} & 2a_{22}A_{22}-\Delta & 2a_{33}A_{23} & \ldots \\ 2a_{11}A_{31} & 2a_{22}A_{32} & 2a_{33}A_{33}-\Delta & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

Three properties of it are established, namely,

(1) *The superadjugate of any determinant* Δ *is equal to* $\Delta^{n-1}\cdot\overline{\Delta}$, *where* $\overline{\Delta}$ *is what* Δ *becomes on changing the sign of all the non-diagonal elements.*

(2) The number of 2-line minors of Δ that are independent of the diagonal elements is $3C_{n,\,4}$.

(3) The superadjugate of Δ will equal Δ^n when and only when

$$\Sigma(\,|a_{11}\, a_{22}\, a_{33}|_0 \cdot a_{44}\, a_{55} \ldots) + \Sigma(\,|a_{11}\, a_{22}\, a_{33}\, a_{44}\, a_{55}|_0 \cdot a_{66} \ldots) + \ldots = 0,$$

where the subscript zero is used to indicate that the determinant to which it is attached has all its diagonal elements changed to 0.

CECIONI, F. (1909/5, 6)

[Sulla caratteristica del prodotto di due matrici. *Periodico di Mat.*, xxiv. pp. 253–265.]

This is essentially the same subject as we have seen dealt with twenty years before under the title 'nullity of a product', the expositor then being Weyr (*v.* above, pp. 3–4). Of more importance, however, is the fact that it now receives fuller treatment than hitherto, and indeed fuller than throughout the period which we have at present under view (*v.* below, p. 57).

MUIR, T. (1909[15]/9)

[Borchardt's form of the eliminant of two equations of the n^{th} degree. *Transac. R. Soc. S. Africa*, i. pp. 447–452.]

Incidentally there is here established the theorem that *If an* m-*by*-(m + 1) *array be such that the sum of every one of its rows vanishes, then the* m-*line determinants of the array, when taken alternately positive and negative, are equal to one another*: and there is thence deduced a generalization of Borchardt's theorem of 1859 (*Hist.*, ii. p. 150), namely, *Any determinant, which has the sum of every row equal to zero, has in the case of every row the same cofactor for every element of the row.*

GRIEND, J. v. d. (1909): BUTAVAND, F. (1911)

[Vraagstuk 103. (Problem of moves on a 4-by-4 draughtboard.) *Wiskundige Opgaven*, x. pp. 241–250.]

[Sur la représentation des déterminants par des systèmes articulés. *L'Enseignement Math.*, xiii. pp. 354–361.]

Curious, but of little importance.

HAYASHI, T. (1910): MIKAMI, Y. (1910)

[The 'Fukudai' and determinants in Japanese mathematics. *Tôkyô Sûgaku-Buturig. Kizi*, (2) v. pp. 254–271: or, in Italian, *Giornale di Mat.*, l. pp. 193–211.]

[Remark on T. Hayashi's article on the 'Fukudai', (2) v. pp. 392–394.]

The glimpse of determinants gained by the early Japanese mathematicians seems to have been similar to that of Leibnitz (*Hist.*, i. pp. 6–10). The more or less imperfect concept arose with them as with him in connection with elimination: and, like him, they had no name for it and no notation. But whereas Leibnitz came on it when dealing with linear equations which he wrote in the form

$$\left.\begin{aligned} 10 + 11x + 12y &= 0 \\ 20 + 21x + 22y &= 0 \\ 30 + 31x + 32y &= 0 \end{aligned}\right\}$$

the Japanese equations were non-linear, and of the type which would now be written

$$\left.\begin{aligned} a + bx + cx^2 &= 0 \\ d + ex + fx^2 &= 0 \\ g + hx + kx^2 &= 0 \end{aligned}\right\}.$$

Their rule for obtaining the result of elimination was also different from his, being that which Hayashi describes shortly as 'cross-multiplication', the terms being got from diagonals of the square array of coefficients in substantially the same manner as that described by Bonolis and others of modern times (*Hist.*, iv. pp. 17–18, 50, . . .).

Hayashi's interesting paper opens with an introduction (pp. 254–256) describing shortly the mathematical course of the school of Seki (1642–1708). He then gives an account (pp. 256–261) of the contents of Seki's booklet on solving 'Fukudai' problems; and this is followed by an explanation (pp. 262–266) of the mode of solution, two examples being given. The final section (pp. 266–271) is devoted to the rule above referred to for accomplishing elimination, and illustrates it fully up to and

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including the case of five equations. The term 'fukudai' is not explained, and we find it hard to say what a 'fukudai' problem exactly is. An example, however, is the following: *Find the greatest side of a triangle* ABC *in which* $a^3 + b^3$, $b^3 + c^3$ *and the area are given.* In the solution of it elimination from three quadratics is utilized.

As the interest of this is in the main historical, it is important to quote Hayashi's personal opinion as given in his concluding paragraph, namely: "Thus we may conclude that the Japanese mathematicians have made use of determinants since the year 1683, and have been able to expand them by a mechanical method such as Sarrus's for the determinant of the third order."

MORITZ, R. E. (1911/3)

[On the cubes of determinants of the second, third, and higher orders. *Bull. American Math. Soc.*, (2) xviii. pp. 182–189.]

The general result established here is

$$|a_1b_2c_3d_4\ldots z_n|^3\cdot|c_1d_4\ldots z_n|\,|c_2d_4\ldots z_n|\,|c_3d_4\ldots z_n| = \begin{vmatrix} A_1{}^2 & A_2{}^2 & A_3{}^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1{}^2 & B_2{}^2 & B_3{}^2 \end{vmatrix}$$

where $A_1, B_1, \ldots$ are the cofactors of $a_1, b_1, \ldots$ in $|a_1b_2c_3d_4\ldots z_n|$. Apparently the author did not observe that it is merely the extensional of the equality

$$|a_1b_2c_3|^3\cdot c_1c_2c_3 = -\begin{vmatrix} |b_2c_3|^2 & |b_3c_1|^2 & |b_1c_2|^2 \\ |b_2c_3|\,|a_2c_3| & |b_3c_1|\,|a_3c_1| & |b_1c_2|\,|a_1c_2| \\ |a_2c_3|^2 & |a_3c_1|^2 & |a_1c_2|^2 \end{vmatrix},$$

which itself is the complementary of

$$|a_1b_2|\cdot|a_2b_3|\cdot|a_3b_1| = -\begin{vmatrix} a_1{}^2 & a_2{}^2 & a_3{}^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \\ b_1{}^2 & b_2{}^2 & b_3{}^2 \end{vmatrix}$$

(*Hist.*, iii. pp. 132, 170).

FROBENIUS, G. (1911$^{12}/_{1}$)

[Ueber den Rang einer Matrix. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1911, pp. 20–29, 128–129.]

Frobenius here follows on the lines of Ed. Weyr's paper of 1889, to which, by way of introduction to his own, he properly draws marked attention. Starting with Sylvester's theorems on nullity (*Hist.*, iv. p. 18) he puts them into his own notation, namely,

$$\rho(AB) \leqq \rho(A) \text{ and } \leqq \rho(B),$$
$$\rho(A) + \rho(B) \leqq n + \rho(AB),$$

pointing out that they are both included in

$$\rho(AB) + \rho(BC) \leqq \rho(B) + \rho(ABC),$$

and making several interesting deductions. The enunciations of these last he much abbreviates by farther curtailing Weyr's notation for a set of linear equations: for example, by so using y, A, x that

$$y = Ax$$

stands for the set of equations

$$\left.\begin{array}{l} y_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ y_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{array}\right\} .$$

As a worthy instance of such a deduction so expressed there may be suitably selected the theorem: *If* L *and* N *be complete solutions of the equations*

$$LBC = 0 \text{ and } ABN = 0$$

then

$$\rho(ABC) - \rho(AB) - \rho(BC) + \rho(B) = \rho(LBN).$$

This is additionally noteworthy because in the next section (§ 2) there is derived from it a variant of another result of Weyr's, namely: *If the integral functions* f(s) *and* g(s) *be mutually prime, and* n $-$ α *be the rank of* f(A) *and* n $-$ β *the rank of* g(A), *then the rank of* f(A) $\cdot$ g(A) *is* n $-$ α $-$ β.

The remaining sections (§§ 3–5) are directly concerned with invariant factors—which, be it remarked, is the main subject of the paper—and fall to be dealt with later.

MUIR, T. (1911)

[The Theory of Determinants in the Historical Order of Development. Vol. II. The Period 1841–1860. xvi + 475 pp. London.]

The second volume of our History.

MUIR, T. ($1911^5/_6$)

[A fifth list of writings on determinants. *Quarterly Journ. of Math.*, xlii. pp. 343–378.]

This list contains in all 300 titles, 113 belonging to the periods of the four preceding lists and 187 to the new five-year period 1906–1910. In the introduction it is noted as curious that of the 113 belated titles as many as 64 belong to the period of the first list, 7 to that of the second, 22 to that of the third, and 20 to that of the fourth; and that a large number of these, although not containing the word 'determinant' as a guide, are of more importance than some of those that do.

NEUBERG, J. ($1912^1/_1$)

[Sur quelques identités. *Mathesis*, (4) ii. pp. 10–14.]

Merely instances of identities that are specially simple-looking when expressed by means of determinants: for example, the identity

$$\sin\alpha\,\sin(\beta-\gamma)+\sin\beta\,\sin(\gamma-\alpha)+\sin\gamma\,\sin(\alpha-\beta)=0$$

and its self-evident equivalent

$$\begin{vmatrix} \sin\alpha & \sin\alpha & \cos\alpha \\ \sin\beta & \sin\beta & \cos\beta \\ \sin\gamma & \sin\gamma & \cos\gamma \end{vmatrix} = 0.$$

SCHUR, I. ($1912^9/_4$)

[Aufgabe 386. *Archiv d. Math. u. Phys.*, (3) xix. p. 276: xxiv. pp. 369–375.]

The theorem here brought forward is *If all the elements of an* n-*line determinant be independent, only* $n^2 - 2n + 2$ *of the terms are independent.* The only proof given of it is not purely algebraical, the writer (G. Polyá) calling to his aid a species of graph which is sometimes made use of in the theory of quantics, but which in the present instance does not conduce to brevity.

SCHÜRER, F. ($1912^{10}/_5$)

[Ueber die Nullpunkte linearer Aggregate von Funktionen. *Jahresb. d. deutschen Math.-Verein.*, xxi. pp. 88–99: xxiii. pp. 42–48.]

Of the first of these papers the third section (pp. 93–95), headed 'Ein Determinantensatz', is occupied with the already oft-discovered condensation-theorem of Chio (*Hist.*, ii. pp. 79–81).

LANCIA, A. ($1912/_5$): PALOMBY, A. (1915)

[Sulle inversioni nelle permutazioni. *Suppl. al Periodico di Mat.*, xv. pp. 101–103.]

[Sulle inversioni nelle permutazioni. *Periodico di Mat.*, xxxi. p. 45.]

Both of these papers concern the problem of finding how many of the permutations on n things have δ inverted-pairs. The first contribution is avowedly an exposition of the simpler known theorems on the subject, and as such is of value; the second contribution is a rediscovery of a theorem dealt with in §§ 23, 24 of Muir's paper of 1899, and more recently by Carlini (see above).

METZLER, W. H. (1912/$_6$)

[On a determinant, all the elements in a certain rectangle of which have a common factor. *Messenger of Math.*, xlii. p. 112.]

The theorem here established is essentially Whitworth's of 1872 (*Hist.*, iii. p 143.)

DZIOBEK, O. (1913$^{29}/_1$)

[Ueber allgemeine Determinantentransformationen. *Sitzungsb. d. Berliner Math. Ges.*, xii. pp. 76–82.]

The main theorem of this interesting paper on 'transformations' concerns the n-by-$2n$ array whose first n-line minor is that having the 'matrix unity', and whose last n-line minor is a general determinant: for example, when n is 3, the array

$$\begin{Vmatrix} 1 & \cdot & \cdot & a_1 & a_2 & a_3 \\ \cdot & 1 & \cdot & b_1 & b_2 & b_3 \\ \cdot & \cdot & 1 & c_1 & c_2 & c_3 \end{Vmatrix}.$$

The number of the n-line minors of the array is of course $(2n)_n$. In ordinary course they do not all remain of the n^{th} order: those of them in which zero elements occur are naturally reducible, and take the form of minors of the general determinant on the right, every minor of the latter making its appearance, even the minors of order 1 and the conveniently tolerated minor of order 0. After reduction the number of order n is 1, of order $n-1$ is n^2, of order $n-2$ is $(n_2)^2$, and so on, this being in keeping with the known equality

$$1 + n^2 + (n_2)^2 + \ldots + n^2 + 1 = (2n)_n.$$

For example, the twenty entities represented by the 3-line minors of the above 3-by-6 array are in order

$$1;\ c_1,\ c_2,\ c_3,\ -b_1,\ -b_2,\ -b_3,\ |\,b_1c_2\,|,\ b_1c_3\,|,\ |\,b_2c_3\,|;$$
$$a_1,\ a_2,\ a_3,\ -|\,a_1c_2\,|,\ -|\,a_1c_2\,|,\ -|\,a_2c_3\,|,\ |\,a_1b_2\,|,\ |\,a_1b_3\,|,\ |\,a_2b_3\,|;$$
$$|\,a_1b_2c_3\,|.$$

Founding on this and on the known effect (*Hist.*, ii. pp. 33–34) of the multiplication of an n-line determinant by an n-by-$(n+h)$ array the author formulates his theorem: *If the* n*-line minors of any* n*-by-*2n *array be divided in order by the first of them, the resulting* $(2\mathrm{n})_n$ *quotients are viewable as the minors of an* n*-line determinant—in other words, stand mutually related as do the corresponding minors of the* n*-by-*2n *array whose first* n*-line minor has unit matrix.* Thus, to take an example of our own, the 5th, 6th, 11th, 12th, 17th minors of the above specialized 3-by-6 being

$$-b_1,\ -b_2,\ a_1,\ a_2,\ |\,a_1b_2\,|,$$

and these being connected by the equality

$$|\,a_1b_2\,| = a_1b_2 - a_2b_1,$$

the corresponding minors of the array

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{matrix}\right\|$$

when each is divided by $|\,x_1\,y_2\,z_3\,|$ are connected in like manner; so that we have

$$\frac{|\,x_3y_4z_5\,|}{|\,x_1y_2z_3\,|} = -\frac{|\,x_2y_3z_4\,|}{|\,x_1y_2z_3\,|}\cdot\frac{|\,x_1y_3z_5\,|}{|\,x_1y_2z_3\,|} + \frac{|\,x_1y_3z_4\,|}{|\,x_1y_2z_3\,|}\cdot\frac{|\,x_2y_3z_5\,|}{|\,x_1y_2z_3\,|},$$

or

$$|\,x_3y_4z_5\,|\;|\,x_1y_2z_3\,| = -\,|\,x_2y_3z_4\,|\cdot|\,x_1y_3z_5\,| + |\,x_1y_3z_4\,|\cdot|\,x_2y_3z_5\,|$$

which is an extensional of

$$|\,x_4y_5\,|\cdot|\,x_1y_2\,| = -\,|\,x_2y_4\,|\cdot|\,x_1y_5\,| + |\,x_1y_4\,|\cdot|\,x_2y_5\,|$$

the otherwise well-known relation between the two-line minors of a 2-by-4 array. Similarly, as the author points out, the equality

$$|\,a_1b_2c_3\,| = a_1\,|\,b_2c_3\,| - a_2\,|\,b_1c_3\,| + a_3\,|\,b_1c_2\,|$$

leads us to

$$\begin{aligned}|\,x_1y_2z_3\,|\;|\,x_4y_5z_6\,| = |\,x_2y_3z_4\,|\;|\,x_1y_5z_6\,| &- |\,x_2y_3z_5\,|\;|\,x_1y_4z_6\,| \\ &+ |\,x_2y_3z_6\,|\;|\,x_1y_4z_5\,|;\end{aligned}$$

and, as we may add, the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |\, a_1 b_2 c_3 \,|$$

leads us to

$$\begin{Vmatrix} |\, x_2 y_3 z_4 \,| & |\, x_2 y_3 z_5 \,| & |\, x_2 y_3 z_6 \,| \\ |\, x_3 y_1 z_4 \,| & |\, x_3 y_1 z_5 \,| & |\, x_3 y_1 z_6 \,| \\ |\, x_1 y_2 z_4 \,| & |\, x_1 y_2 z_5 \,| & |\, x_1 y_2 z_6 \,| \end{Vmatrix} = |\, x_4 y_5 z_6 \,| \; |\, x_1 y_2 z_3 \,|^2.$$

Instead of this last, which is Bazin's, the author derives Sylvester's outwardly resembling result of the same year (*Hist.*, iv. p. 205).

Of course we may take for the second 3-by-6 array a mere permutation of the first; but as the new minors in this case can only be a permutation of those we already have, the derived equality will naturally be more curious than valuable; for example, if the array be

$$\begin{Vmatrix} a_1 & a_2 & \cdot & 1 & \cdot & a_3 \\ b_1 & b_2 & \cdot & \cdot & 1 & b_3 \\ c_1 & c_2 & 1 & \cdot & \cdot & c_3 \end{Vmatrix}$$

the derived companion result to the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |\, a_1 b_2 c_3 \,|$$

is

$$\begin{vmatrix} b_2 & a_2 & -|\, a_2 b_3 \,| \\ b_1 & a_1 & -|\, a_1 b_3 \,| \\ |\, b_1 c_2 \,| & |\, a_1 c_2 \,| & |\, a_1 b_2 c_3 \,| \end{vmatrix} = c_3 \,|\, a_1 b_2 \,|^2$$

an equality readily verified by the operation

$$\text{row}_3 + c_1\text{row}_1 - c_2\text{row}_2.$$

METZLER, W. H. (1913/1)

[Rectangular arrays. *Rendic. del Circolo Mat.*, (Palermo), xxxvii. pp. 332–340.]

The subject here is the multiplication of oblong arrays. At the outset the author takes us back to the earliest contributions

to the subject—the two famous memoirs which were presented to the French Academy of Sciences on the same day of 1812 by Binet and Cauchy, and which, because of their supreme importance, were made the subject of a separate chapter of over 50 pages in the first volume of our History (*Hist.*, i. pp. 80–131). It may be remembered that the main instance of overlapping in the two memoirs concerned the extended multiplication-theorem, and that, in view of a statement of Cauchy's, a close investigation was made (pp. 122–130) in order to make clear exactly what each writer had contributed to the generalization. In doing so we formulated the widest theorem reached, Binet's, adopting his justifiable assumption that what he had proved to hold in the case of 2-rowed arrays, held generally. Now although Metzler's paper throws light on the whole subject, it is this particular theorem which forms the starting point of his work. First of all, he generalizes it, so as to have the number of arrays in the one sum not necessarily the same as the number in the other. As thus widened the theorem may be roughly described as giving an expression for the product of two sums of m-by-n arrays in the form of a sum of products of pairs of sums of m-line minors of the arrays. For example, if m is 2 and n is 3, and the number of arrays in the one sum is 4 and in the other 5, we have

$$\left\{ \left\| \begin{matrix} a_1a_2a_3 \\ b_1b_2b_3 \end{matrix} \right\| + \left\| \begin{matrix} c_1c_2c_3 \\ d_1d_2d_3 \end{matrix} \right\| + \left\| \begin{matrix} e_1e_2e_3 \\ f_1f_2f_3 \end{matrix} \right\| + \left\| \begin{matrix} g_1g_2g_3 \\ h_1h_2h_3 \end{matrix} \right\| \right\}$$

$$\cdot \left\{ \left\| \begin{matrix} \kappa_1\kappa_2\kappa_3 \\ \lambda_1\lambda_2\lambda_3 \end{matrix} \right\| + \left\| \begin{matrix} \mu_1\mu_2\mu_3 \\ \nu_1\nu_2\nu_3 \end{matrix} \right\| + \left\| \begin{matrix} \rho_1\rho_2\rho_3 \\ \sigma_1\sigma_2\sigma_3 \end{matrix} \right\| + \left\| \begin{matrix} \phi_1\phi_2\phi_3 \\ \chi_1\chi_2\chi_3 \end{matrix} \right\| + \left\| \begin{matrix} \psi_1\psi_2\psi_3 \\ \omega_1\omega_2\omega_3 \end{matrix} \right\| \right\}$$

$$\begin{aligned} = & \{ |a_1b_2| + |c_1d_2| + |e_1f_2| + |g_1h_2| \} \\ & \quad \cdot \{ |\kappa_1\lambda_2| + |\mu_1\nu_2| + |\rho_1\sigma_2| + |\phi_1\chi_2| + |\psi_1\omega_2| \} \\ & + \{ |a_1b_3| + |c_1d_3| + |e_1f_3| + |g_1h_3| \} \\ & \quad \cdot \{ |\kappa_1\lambda_3| + |\mu_1\nu_3| + |\rho_1\sigma_3| + |\phi_1\chi_3| + |\psi_1\omega_3| \} \\ & + \{ |a_2b_3| + |c_2d_3| + |e_2f_3| + |g_2h_3| \} \\ & \quad \cdot \{ |\kappa_2\lambda_3| + |\mu_2\nu_3| + |\rho_2\sigma_3| + |\phi_2\chi_3| + |\psi_2\omega_3| \} \\ = & \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Using the notation of his paper of 1897 (*Hist.*, iv. pp. 220–222)

Metzler with ease to himself states the general theorem in a single line. He then deduces from it as a special case Cauchy's corresponding theorem, thus supporting Muir's conclusion as to the nature of the relation between the two. Other deductions of a quite general character are made, and a still greater number are got by specializing, the more interesting of the latter being due to making the second sum of arrays identical with the first. In this way special subjects previously discussed by others, and even determinants of special form like the Compound and the Orthogonant come in for consideration and advancement.

POLYÁ, G. ($1913^3/_1$)

[Aufgabe 424. *Archiv d. Math. u. Phys.*, (3) xx. p. 271.]

The proposition here set for proof is that by no mere alteration of signs in the elements of

$$\overset{+}{|}\, a_{rs} \,\overset{+}{|}_n$$

can it be made equal to $|\, a_{rs} \,|_n$ when $n > 2$.

BOTTASSO, M. ($1913^6/_1$)

[Sui sistemi di equazioni ottenuti da un determinante simmetrico di forme in più serie di variabili. *Rendic. . . . Ist. Lombardo* . . ., (2) xlvi. pp. 88–108.]

As a foundation for his main subject the author here first lays down three theorems connected with general evanescent arrays, following them up with other three dealing with square arrays that are axisymmetric. It is the former three that now concern us. The first theorem of all is in effect: *If any* m-*by*-n *array be evanescent, and at the same time the array got from it by deleting both its last row and its last column, then either* (1) *the array got by deleting only the last row is evanescent, or* (2) *the array got by deleting only the last column is not merely evanescent but has all its* (m − 1)-*line minors equal to* 0. The important case where $m = n$ we may state for ourselves separately: *If any determinant,* D *say, vanishes, and at the same time the cofactor of its last element,*

then one of two arrays must also vanish, namely; either the array got from D *by deleting its last row or the array got by deleting its last column.* A proof of this is given, with an indication of how to extend it. The two other theorems we shall condense in statement. The one is:

$$\textit{If}\ \left\|\begin{matrix}1, 2, \ldots, m & \\ 1, 2, \ldots\ldots, & n\end{matrix}\right\| \equiv 0 \equiv \left\|\begin{matrix}1, 2, \ldots, m-1, & \\ 1, 2, \ldots\ldots\ldots\ldots, & p\end{matrix}\right\|\ n>p>m-2,$$

$$\textit{then either}\qquad \left\|\begin{matrix}1, 2, \ldots, m-1 & \\ 1, 2, \ldots\ldots\ldots\ldots, & n\end{matrix}\right\| \equiv 0,$$

$$\textit{or, all the}\ (\mathrm{m}-1)\textit{-line minors of}\left\|\begin{matrix}1, 2, \ldots, m & \\ 1, 2, \ldots\ldots, & p\end{matrix}\right\|\textit{vanish.}$$

The other is:

$$\textit{If}\qquad \left\|\begin{matrix}1, 2, \ldots, m & \\ 2, \ldots\ldots, & n\end{matrix}\right\| \equiv 0 = \left|\begin{matrix}1, 2, \ldots, m \\ 1, 2, \ldots, m\end{matrix}\right|,$$

$$\textit{then}\qquad \textit{either}\left\|\begin{matrix}1, 2, \ldots, m & \\ 1, 2, \ldots\ldots, & n\end{matrix}\right\| \equiv 0$$

$$\textit{or}\qquad \left\|\begin{matrix}1, 2, \ldots, m & \\ 2, \ldots\ldots, & n\end{matrix}\right\| \equiv 0.$$

A reinvestigation of the three on different lines would be of value, especially if throughout it all previous related theorems were kept in view.

MUIR, T. ($1913^8/_1$)

[Note on an overlooked theorem regarding the product of two determinants of different orders. *Transac. R. Soc. S. Africa*, iii. pp. 271–273.]

The overlooked theorem in question is that enunciated by Simonnet in 1879 but not proved. Here it is recast, widened and proved by Muir on the analogy of the so-called Sylvester's theorem of 1851 (*Hist.*, ii. pp. 61–62, iv. p. 81). As now viewed it expresses the product of two determinants of the p^{th} and q^{th} orders ($p > q$) as a sum of products of two determinants of the $(p-a)^{\text{th}}$ and $(q+a)^{\text{th}}$ orders.

MUIRHEAD, R. F. (1913¹/₅)

[Two theorems on determinants, and their application . . . *Math. Notes* (of Edinburgh Math. Soc.), i. pp. 151–153.]

For the third order the second theorem is

$$\begin{vmatrix} a_1 + \lambda(a_2 - a_1) & a_2 + \lambda(a_2 - a_1) & a_3 \\ b_1 + \lambda(b_2 - b_1) & b_2 + \lambda(b_2 - b_1) & b_3 \\ c_1 + \lambda(c_2 - c_1) & c_2 + \lambda(c_2 - c_1) & c_3 \end{vmatrix} = | a_1 b_2 c_3 |.$$

MÉTROD, G. (1913¹/₇): ROSS, C. M. (1913¹/₁₀)

[Question 4245. *L'Intermédiaire des Math.*, xx. p. 147.]

[Question 17602. *Educ. Times*, lxvi. p. 434: or *Math. from Educ. Times*, (2) xxviii. pp. 46–47: or *Mathesis*, xxxix. pp. 473–474.]

The first question here concerns the expression of the m^{th} power of any 2-line determinant as a 2-line determinant. The treatment of it is quite ineffective, the law of formation of the elements of the product of m determinants being overlooked (*Hist.*, iv. pp. 27–28).

The other concerns a case of the equality so effectively treated by Metzler in 1899 (*Hist.*, iv. pp. 495–496).*

STEPHANOS, C. (1913): FATERSON, L. (1913)

[Sur une propriété caractéristique des déterminants. *Annali di Mat.*, xxi. pp. 233–236.]

[O warunku koniecznym i ostatecznym by wyznacznik równa się zeru. *Wektor* (Warszawa), iii. pp. 116–119.]

The question here raised by Stephanos is whether there may not be other functions than determinants that have the same law

* S. Pollard's question 17783, which appeared a few months later than this in the same serial, concerns a vanishing 3-line determinant, but through incorrectness of printing or other cause is unintelligible.

of multiplication: and the result reached is that *the only functions of* n^2 *variables* x_{ij}, *say the function* $f(x)$, *which are such that*

$$f(x)\cdot f(y) = f(z) \quad \textit{when} \quad z_{ik} = \Sigma x_{ik} y_{kj}$$

and such that they can be differentiated with respect to all their variables, are those which are a power of $|\, x_{ij} \,|_n$.

The other subject is of old standing—a test for a null determinant (cf. *Hist.*, ii. pp. 94–95).

METZLER, W. H. (1913/$_{11}$)

[The rank of a matrix. *Annals of Math.*, xv. pp. 161–165.]

The main theorem here established is an extension of Kronecker's of 1864 (*Hist.*, iii. pp. 14, 25), being as follows: *If in an array there be a* k-*line non-vanishing minor, while all the* (k + h)-*line minors which include the said minor do vanish, then every other* (k + h)-*line minor must vanish also.* It leads up to two allied theorems, one regarding axisymmetric and one regarding zero-axial skew determinants.

CARLINI, L. (1913)

[Sopra un simbolo operativo la cui teoria presenta analogie con quella dei determinanti. 15 pp. Udine.]

The actual contents of this paper and the contents as indicated in the title are seriously discrepant, the operational symbol given prominence to in the title being very imperfectly dealt with in the paper, and the matter most fully discussed in the paper being not referred to in the title at all. Taking these points in reverse order we have thus to note first that the author is mainly occupied with the consideration of those permutations which have the same given number of inverted-pairs—a subject all the more undesirable to exclude from the chance of being indexed because other papers had already been devoted to it, namely, Bourget's paper of 1871 (*Hist.*, iii. p. 33), Muir's of 1899 (*Hist.*, iv. pp. 72–73), and one by Carlini himself in 1903 (see above).

The number in question when there are n things permuted

and when the given fixed number of inverted-pairs is r, the author denotes by $A_{n,r}$, and he states its elementary properties to be

$$A_{rr} = A_{n,\frac{1}{2}n(n-1)-r},$$
$$A_{n,\frac{1}{2}n(n-1)} = 1,$$
$$A_{n,0} + A_{n,1} + A_{n,2} + \ldots + A_{n,\frac{1}{2}n(n-1)} = n!$$

If, however, the reader will make allowance for differences of notation, and note especially that the symbol $A_{n,r}$ had previously been written $V_{n,r}$ he will have no difficulty in verifying the fact that the properties in question are rediscoveries. More interesting for comparison with previous work is the fresh mode of utilizing the concept of the 'generating function' in investigating $A_{n,r}$.

KOCH, H. v. (1913)

[Ueber das Nichtverschwinden einer Determinante, nebst Bemerkungen über Systeme unendlich vieler linearer Gleichungen. *Jahresb. d. deutschen Math.-Verein.*, xxii. pp. 285–291.]

The basic result here on non-evanescence is spoken of as Hadamard's of 1903, but is in reality Lévy's of 1881 and Desplanques' of 1886 (see above, pp. 33–34). The paper opens with an interesting but rather over-condensed proof of what is viewed to be an equivalent theorem, namely, *If in the determinant*

$$\begin{vmatrix} 0 & -b_{12} & -b_{13} & \ldots \\ -b_{21} & 0 & -b_{23} & \ldots \\ -b_{31} & -b_{32} & 0 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n$$

the sum of the moduli of the elements of each row is not greater than a quantity ϵ *that is less than* 1, *the determinant cannot vanish.*

The initial step taken is

$$\begin{vmatrix} 1 & -b_{12} & -b_{13} & \ldots \\ -b_{21} & 1 & -b_{23} & \ldots \\ -b_{31} & -b_{32} & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n \cdot \begin{vmatrix} 1 & b_{12} & b_{13} & \ldots \\ b_{21} & 1 & b_{23} & \ldots \\ b_{31} & b_{32} & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n$$

$$= \begin{vmatrix} 1-(b_{12}b_{21}+b_{13}b_{31}) & -b_{13}b_{32} & -b_{12}b_{23} \\ -b_{23}b_{31} & 1-(b_{21}b_{12}+b_{23}b_{32}) & -b_{21}b_{13} \\ -b_{32}b_{21} & -b_{31}b_{12} & 1-(b_{31}b_{13}+b_{32}b_{23}) \end{vmatrix}^{*},$$

or, as it is abbreviated,

$$D(b) \cdot D(-b) = D(b^{(2)});$$

whence, after repetition of similar multiplications, there is obtained

$$D(b) \cdot D(-b) \cdot D(b^{(2)}) \cdot D(-b^{(2)}) \ldots D(-b^{(2^p-1)}) = D(b^{(2^p)}).$$

The next step is to show that

$$\lim_{\nu=\infty} b^{(\nu)}_{\iota\kappa} = 0,$$

an immediate consequence of which is that the infinite product of D's is 1 and that therefore no single one of the D's can vanish.

The subject is continued in regard to determinants of infinite order, and thereafter an application made as indicated in the title.

TOCCHI, L. (1914/1): MARIARES, F. (1914¹/₇)

[Generalizzazione d'un teorema sui determinanti. *Periodico di Mat.* Anno xxix. pp. 53–58.]

[Tres teoremas sobre determinantes. *Revista de la Soc. Mat. Españ.*, iii. pp. 303–306.]

The first result established by Tocchi is that *If an* m-*by*-n *array be evanescent, the* (m — 1)*-line minors of any* m — 1 *rows are proportional to the corresponding minors of any other* m — 1 *rows*;

* This equality it is well to recognize as being included in one with which we are familiar in the theory of latent roots, namely,

$$\begin{vmatrix} a_1-x & a_2 & a_3 & \ldots \\ b_1 & b_2-x & b_3 & \ldots \\ c_1 & c_2 & c_3-x & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \cdot \begin{vmatrix} a_1+x & a_2 & a_3 & \ldots \\ b_1 & b_2+x & b_3 & \ldots \\ c_1 & c_2 & c_3+x & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$\begin{vmatrix} \text{row}_1\cdot\text{col}_1 - x^2 & \text{row}_1\cdot\text{col}_2 & \text{row}_1\cdot\text{col}_3 & \ldots \\ \text{row}_2\cdot\text{col}_1 & \text{row}_2\cdot\text{col}_2 - x^2 & \text{row}_2\cdot\text{col}_3 & \ldots \\ \text{row}_3\cdot\text{col}_1 & \text{row}_3\cdot\text{col}_2 & \text{row}_3\cdot\text{col}_3 - x^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

where row_k stands for the k^{th} row of $|a_1b_2c_3|$.

and then with this as a lemma there is proved with equal care and fullness the generalized theorem in question, namely, *If a determinant be of non-zero rank* r, *then the* r-*line minors of any* r *rows are proportional to the corresponding minors of any other* r *rows*. No reference is made to Frobenius' related work of 1876 (*Hist.*, iii. pp. 62–64), to Capelli-Garbieri (§ 371) of 1886, or to Garbieri (§ 15) of 1891 (*Hist.*, iv. pp. 102–103, 105–106).

The theorems of the second paper are quite elementary.

MUIR, T. ($1914^{20}/_1$)

[Note on the sum of the equigrade minors of a determinant. *Messenger of Math.*, xliii. pp. 177–184.]

The problem here set is to affix an r-line border to an n-line determinant, so as to obtain an equivalent for the sum of the $(n - r)$-line minors of the determinant, as is known to be possible when r is 1 (*Hist.*, iii. pp. 11, 13), for example,

$$\begin{vmatrix} . & 1 & 1 & 1 \\ -1 & a_1 & a_2 & a_3 \\ -1 & b_1 & b_2 & b_3 \\ -1 & c_1 & c_2 & c_3 \end{vmatrix} = A_1+A_2+A_3+B_1+B_2+B_3+C_1+C_2+C_3.$$

It is first shown to be soluble for the case $n = 3$, $r = 2$, the result being

$$\begin{vmatrix} . & . & -1 & . & 1 \\ . & . & . & -1 & 1 \\ 1 & . & a_1 & a_2 & a_3 \\ . & 1 & b_1 & b_2 & b_3 \\ -1 & -1 & c_1 & c_2 & c_3 \end{vmatrix} = a_1+a_2+a_3+b_1+b_2+b_3+c_1+c_2+c_3.$$

Next it is proved, on the other hand, that values cannot be found for the x's and y's in

$$\begin{vmatrix} . & . & x_1 & x_2 & x_3 & x_4 \\ . & . & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & a_1 & a_2 & a_3 & a_4 \\ x_2 & y_2 & b_1 & b_2 & b_3 & b_4 \\ x_3 & y_3 & c_1 & c_2 & c_3 & c_4 \\ x_4 & y_4 & d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

so as to make it equal to the sum of the two-line minors of $|a_1b_2c_3d_4|$; and what follows thereafter renders it probable that the only other soluble case is where $n = n$ and $r = n - 1$, the border then being of the form

$$\begin{array}{cccccc} -1 & \cdot & \cdot & \cdots & \cdot & 1 \\ \cdot & -1 & \cdot & \cdots & \cdot & 1 \\ \cdot & \cdot & -1 & \cdots & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & -1 & 1. \end{array}$$

For example

$$\begin{vmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \\ 1 & \cdot & \cdot & a_1 & a_2 & a_3 & a_4 \\ \cdot & 1 & \cdot & b_1 & b_2 & b_3 & b_4 \\ \cdot & \cdot & 1 & c_1 & c_2 & c_3 & c_4 \\ -1 & -1 & -1 & d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{cases} a_1 + a_2 + a_3 + a_4 \\ + b_1 + b_2 + b_3 + b_4 \\ + c_1 + c_2 + c_3 + c_4 \\ + d_1 + d_2 + d_3 + d_4. \end{cases}$$

In every case, it will be observed, the bordering gnomon is skew.

MUIR, T. (1914[20]/[2])

[The determinant of the sum of a square matrix and its conjugate. *Messenger of Math.*, xliii. pp. 184–192.]

This peculiar form of determinant,* spoken of here as the 'duplicant', seems to have been first thought of by Hunyady in 1882 (*Hist.*, iv. p. 209). By expressing it as a sum of determinants with monomial elements there comes at once the result: *The duplicant of any* n*-line determinant when* n *is odd is equal to twice the sum of* 2^{n-1} *determinants*: for example, when n is 3 the duplicant equals

$$2\{\,|123| + |123'| + |12'3| + |1'23|\,\},$$

where 1, 2, 3 are the columns of the basic determinant, and $1', 2', 3'$ are those of the conjugate. When n is 4, the corresponding development is

$$2\{\,|1234| + \Sigma\,|1234'|\,\} + \Sigma\,|123'4'|\,.$$

* Though peculiar and bearing a special name, there is a certain appropriateness in assigning it to Chap. I, unless its basic matrix be specialized.

Following this comes a development in which the two sets of columns are kept separate, for example,

$$\text{dupl.}\,|\,a_1b_2c_3d_4\,| = 2\{\,|\,a_1b_2c_3d_4\,| + \Sigma(A_1, B_1, C_1, D_1 \between a_1, a_2, a_3, a_4)\}$$
$$+\ \Sigma\{\,|\,a_1b_2\,|\cdot\text{conj. compl.}\,|\,a_1b_2\,|\,\},$$

where conj. compl. $|\,a_1b_2\,| = \text{conj.}\,|\,c_3d_4\,| = |\,c_3d_4\,|$. This leads to a rediscovery of Hunyady's neglected theorem which is now proved for the first time. The rest of the paper is occupied with the consideration of bordered duplicants.

PASCAL, A. (1914/9)

[Sopra i minori del determinante generalizzato di Scholtz-Hunyady. *Giornale di Mat.*, lii. pp. 301–304.]

This paper follows up Cazzaniga's of 1900, the term 'Scholtz-Hunyady' being a well-meant substitute for the former 'Hunyady', and the so-called generalized determinant being merely the determinant in its original form of the year 1851,*—that is to say, where the elements are of the degree r. The only difference in the process lies in the fact that the number of equations in the derived set is $(n + r - 1)_r$ instead of $(n + 2 - 1)_2$: as before, the basic set and the derived set are solved and the solutions brought into relation. What we have to take note of is the correspondingly generalized result, namely, that *the factor common to all the primary minors of the derived determinant is that power of the basic determinant whose index is* $(\mathrm{n} + \mathrm{r} - 1)_r - \mathrm{r}$, and that the cofactors appear as sums of products of primary minors of the basic determinant.

To help in the visualization of this we give another instance of our summarizing theorem exemplified under Cazzaniga: *The basic determinant being* $|\,a_1b_2\,|$, *and Schläfli's derived determinant when* r *is* 3 *being consequently*

$$\begin{vmatrix} a_1^3 & 3a_1^2a_2 & 3a_1a_2^2 & a_2^3 \\ a_1^2b_1 & a_1^2b_2 + 2a_1a_2b_1 & a_2^2b_1 + 2a_1a_2b_2 & a_2^2b_2 \\ a_1b_1^2 & b_1^2a_2 + 2b_1b_2a_1 & b_2^2a_1 + 2b_1b_2a_2 & a_2b_2^2 \\ b_1^3 & 3b_1^2b_2 & 3b_1b_2^2 & b_2^3 \end{vmatrix},$$

* See footnote under Whittaker (1916) in Chap. on Orthogonants.

the adjugate of the latter determinant is equal to

$$|a_1b_2|^{3\cdot 4}\begin{vmatrix} A_1^3 & 3A_1^2A_2 & 3A_1A_2^2 & A_2^3 \\ A_1^2B_1 & A_1^2B_2+2A_1A_2B_1 & A_2^2B_1+2A_1A_2B_2 & A_2^2B_2 \\ A_1B_1^2 & B_1^2A_2+2B_1B_2A_1 & B_2^2A_1+2B_1B_2A_2 & A_2B_2^2 \\ B_1^3 & 3B_1^2B_2 & 3B_1B_2^2 & B_2^3 \end{vmatrix}$$

where $|A_1B_2|$ *is the adjugate of* $|a_1b_2|$.

KNESER, A. (1914)

[Zur Theorie der Determinanten. *Schwarz-Festschrift*, pp. 177–191.]

The author here starts with the rather doubtful assertion that the theorems usually proved with the help of determinants may be separated into two groups—one in which the subject-matter of the theorem is independent of the determinant concept, and one in which it is not. In support of this and to make clear the difference between the two he first selects four elementary theorems regarding linear forms and linear equations, and gives demonstrations of them without making use of determinants (pp. 177–183); and then for his complementary illustration he brings forward (pp. 183–187) Kummer's non-determinant investigation of Lagrange's cubic equation (*Hist.*, ii. pp. 295–296) as followed up on a different line by Jacobi and Borchardt (*Hist.*, ii. pp. 296–302). Through his treatment of this latter illustration he is led on to the subject of integral equations,* with the result that we have an unexpected addendum (pp. 187–191) of more interest than the main body of the paper.

SCHUCHMANN, H. (1914/5)

[Die Determinante in der japanischen Mathematik und die Erweiterung der Sarrus'schen Verfahrens für die Auflösung der Determinanten dritten Grades auf Determinanten n-ten Grades. *Math.-Naturw. Blätter*, xi. pp. 65–68, 86–89.]

This is a careful exposition of the contents of Hayashi's paper of 1910. Anything additional is due to the fact that the author

* The paper is thus seen to have a bearing on three of our later chapters—those on Linear Equations, Orthogonants, and Wronskians.

attaches more than the usual importance to the so-called Japanese method of development, and explains it with considerable fullness for the case of the fourth order.

KOJIMA, T. (1914, 1917)

[On a theorem of Hadamard's and its application. *Tôhoku Math. Journ.*, v. pp. 54–60.]

[On the limits of the roots of an algebraical equation. *Tôhoku Math. Journ.*, xi. pp. 119–127.]

The determinantal theorem here utilized is called by the author Hadamard's of 1903; but, as we have already seen, it is of much earlier date (see above, p. 33). In the first paper, before applying the theorem, a fresh proof of it is given, based on the theory of linear equations. The second paper merely repeats and improves the application made in the first.

PASCAL, A. ($1915/_1$)

[Sui determinanti ottenuti da un altro operando una medesima trasformazione lineare sugli elementi di una o più colonne. *Giornale di Mat.*, liii. pp. 35–42.]

The first part of this paper (§§ 1–3) continues the consideration of a subject previously dealt with by Bagnera and E. Pascal (*Hist.*, iv. pp. 33–34, 63). The opening theorem may be formulated thus: D *being any* n-*line determinant and* Δ *its adjugate*; D_h *the determinant got from* D *by substituting for each element of the* h^{th} *column a linear function of the elements of that column, and* Δ_h *the determinant similarly got from* Δ: *then, if the determinant of the coefficients of the substitution in the one case be the conjugate of the determinant in the other,*

$$\Delta_h = D_h \cdot D^{n-2}.$$

For example, when n is 3 and h is 2, we have

$$\begin{vmatrix} A_1 & (A_2, B_2, C_2 \between \alpha_1, \beta_1, \gamma_1) & A_3 \\ B_1 & (A_2, B_2, C_2 \between \alpha_2, \beta_2, \gamma_2) & B_3 \\ C_1 & (A_2, B_2, C_2 \between \alpha_3, \beta_3, \gamma_3) & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & (a_2, b_2, c_2 \between \alpha_1, \alpha_2, \alpha_3) & a_3 \\ b_1 & (a_2, b_2, c_2 \between \beta_1, \beta_2, \beta_3) & b_3 \\ c_1 & (a_2, b_2, c_2 \between \gamma_1, \gamma_2, \gamma_3) & c_3 \end{vmatrix} \cdot | a_1 b_2 c_3 |.$$

When more columns than one in D and Δ are subjected to substitution, a quite analogous result holds, the work of establishing it, however, being of course considerably more complicated.

In the second part of the paper (§§ 4–6) determinants obtained from D by substitution are still under consideration, but the problems solved are different in kind, the aim no longer being the discovery of a relationship but the effecting of a summation. Thus, to take the first and simplest example, if not all the n elements of the h^{th} column are at one time to be affected but only k of them, there arises not the sole determinant D_h but a series of $(n)_k$ determinants, and the sum of the series is shown to be

$$(n-1)_k D + (n-1)_{k-1} D_h.$$

NANSON, E. J. ($1915^1/_2$)

[Question 17932. *Educ. Times*, lxviii. p. 78.]

The result here announced for proof is in effect that at most $\frac{1}{6}m(m-1)(3n-m-1)$ minors of an m-by-n array have to be examined in order to ascertain the non-zero rank of the array.

DECKER, F. F. ($1915/_4$)

[On the order of a restricted system of equations. *American Journ. of Math.*, xxxvii. pp. 159–178.]

As auxiliary to the main business of this paper two sections (pp. 164–170) are devoted to establishing Cayley's theorem of 1843 regarding a null array, no mention, however, being made of the original author. It is first shown that *the* m-*line minors of an* m-*by*-(m + r) *array are linearly expressible in terms of the first* r + 1 *of them.* The proof, which is gradational in form, occupies four pages, the cases $r = 1$, $r = 2$, $r = 3$ being dealt with at full length. Apparently the evidence involved in it is considered

equivalent to showing at the same time that *the number of linearly independent* m-*line minors in the array is* r + 1. The problem of obtaining the actual expressions referred to in the first of the two statements is also formally solved, there being thus given in a sense a second proof. These expressions, however, are not observed by the author to be merely cases of the so-called Sylvester theorem for expressing the product of two determinants as a sum of like products. They resemble Muir's of 1891 (*Hist.*, iv. pp. 47–48), but differ in that the multipliers of the first $r+1$ minors are of the $(m-1)^{\text{th}}$ order and do not contain any arbitrary elements. Thus if the array be

$$\left\|\begin{matrix} a & b & c & d & & \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\|,$$

and it is desired to obtain an expression linearly connecting $|a_1b_2c_3d_4|$ with the first three 4-line minors of the array, the relation of 1891 is

$$|a_1b_2c_3d_7|\,|a_1b_2c_4d_5| = |a_1b_2c_4d_7|\,|a_1b_2c_3d_5| - |a_1b_2c_5d_7|\,|a_1b_2c_3d_4|$$

and the relation of 1915 is

$$-|b_1c_2d_3|\,|a_1b_2c_4d_5| = |b_1c_2d_4|\,|a_1b_2c_3d_5| - |b_1c_2d_5|\,|a_1b_2c_3d_4|.$$

The latter is obtainable from the former by putting the arbitrary elements a_7, b_7, c_7, d_7 equal to 1, 0, 0, 0 respectively. Both lead more or less readily to the condition attached by Rahusen to the theorem under discussion, namely, that

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\| \neq 0.$$

(*Hist.*, iv. pp. 40–41).

The main subject of the paper as indicated in the title has been of interest for a long period, and a considerable number of writings have been occupied with it, all of them to a greater or less degree making use of determinants, but none of them save Decker's stepping aside to discuss the instrument used. The titles of the more relevant of these writings are included in the List given at the end of this chapter.

BOEHM, K. (1915[1]/[5])

[Ueber einen Determinantensatz, in welchem das Multiplicationstheorem als besonderer Fall enthalten ist. *Crelle's Journ.*, cxlv. pp. 250–253.]

The interesting theorem here established we may enunciate for ourselves thus: *If* A *and* B *be any* n-*line determinants whose product is* P, *and every possible determinant be taken that is formable by replacing* m *rows of* A *by the corresponding rows of* P, *then the sum of these* $(\mathrm{n})_m$ *determinants is equal to the product of* A *by the sum of the* m-*line coaxial minors of* B: for example, when $n = 3$ and $m = 2$, the basic determinants being $|a_1 b_2 c_3|$, $|x_1 y_2 z_3|$, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \Sigma bx & \Sigma by & \Sigma bz \\ \Sigma cx & \Sigma cy & \Sigma cz \end{vmatrix} + \begin{vmatrix} \Sigma ax & \Sigma ay & \Sigma az \\ b_1 & b_2 & b_3 \\ \Sigma cx & \Sigma cy & \Sigma cz \end{vmatrix} + \begin{vmatrix} \Sigma ax & \Sigma ay & \Sigma az \\ \Sigma bx & \Sigma by & \Sigma bz \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= |a_1 b_2 c_3| \cdot \mathrm{saxm}_2 \, |x_1 y_2 z_3| \, ,$$

where $\Sigma ax, \ldots$ stand for $a_1x_1 + a_2x_2 + a_3x_3, \ldots$, and saxm_2 for 'the sum of the 2-line coaxial minors of'. The author says that the theorem was suggested to him by two simple cases found in joint writings of Burali-Forti and Marcolongo.* The demonstration given is less direct than it might be. By using Laplace's expansion-theorem and the multiplication-theorem the series of derived determinants can be expressed as an aggregate of $\{(n)_m\}^2$ terms, each one having an m-line minor of B as a factor and either A or 0 for the cofactor according as the said minor is coaxial or not.

It may be worth adding that, by Deruyts' theorem of 1882 (*Hist.*, iv. p. 39), the result would be the same if in the formation of the determinants derived from A and P columns were used instead of rows: and that the compound determinant with elements constructed after the manner of the said derived determinants is equal to

$$A^{(n)_m} B^{(n-1)_{m-1}}.$$

(*Hist.*, iii. p. 198).

* *Omografie Vettoriale*, § 2, Torino, 1909: *Analyse Vectorielle Générale*, i. § 8: Pavie, 1912.

ROSS, C. M. ($1915^1/_5$)

[Question 17999. *Educ. Times*, lxviii. p. 203. *Math. Qu Sol.*, i. pp. 93–94.]

An interesting example of an evanescent array rows and $2n$ columns. When n is 4 the equality is

$$\left\|\begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & . & . & . & . \\ a_1 & a_2 & a_3 & . & . & . & . & x_4 \\ a_1 & a_2 & . & a_4 & . & . & x_3 & . \\ a_1 & . & a_3 & a_4 & . & x_2 & . & . \\ . & a_2 & a_3 & a_4 & x_1 & . & . & . \\ . & . & . & . & x_1 & x_2 & x_3 & x_4 \end{array}\right\| = 0.$$

It is based on the fact that the 5th and 6th rows can both be changed into

$$4a_1 \; 4a_2 \; 4a_3 \; 4a_4 \; x_1 \; x_2 \; x_3 \; x_4.$$

KÖNIG, D. ($1915/_{10}$)

[Ueber Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Annalen*, lxxvii. pp. 453–465; or, in Magyar, *Math. és termesz. értesitö* (Budapest), xxxiv. pp. 104–119.]

'Graph', as here used, recalls Schur's paper of 1912. It is a figure consisting of any number of points of which certain pairs are joined. The section (pp. 457–459) dealing with the application to determinants results in two theorems of a peculiar type: (1) *If each row and each column of a determinant have the same positive sum, and the elements be non-negative, there is at least one non-zero term in the development.* (2) *If the number of non-zero elements in each row and each column be* k, *there are at least* k *non-zero terms in the development.*

MUIR, T. ($1915^3/_{11}$)

[Note on the so-called Vahlen relations between the minors of a matrix. *Transac. R. Soc. S. Africa*, v. pp. 695–701.]

This is a critical examination of Vahlen's paper of 1893 (*Hist.*, iv. pp. 55–56), making clear at the same time how little ground there is for connecting his name with the subject. As we have

seen in the course of our chronicle, the papers really deserving of mention are Sylvester's and Bazin's of 1851 (*Hist.*, ii. pp. 61–62, 206–208), d'Ovidio's of 1877, and Rubini's of 1878 (*Hist.*, iii. pp. 68, 200).

Two incidental results may be briefly noted: (1) that *any minor of a Bazin's compound determinant is itself a determinant of the same kind*: and (2) that since (*Hist.*, iv. p. 273)

$$\left|\begin{array}{ccc} |a_1b_2| & |a_1c_2| & |a_1d_2| \\ & |b_1c_2| & |b_1d_2| \\ & & |c_1d_2| \end{array}\right| = 0,$$

we may with much convenience write five such identities in the form

$$\left\|\begin{array}{cccc} |a_1b_2| & |a_1c_2| & |a_1d_2| & |a_1e_2| \\ & |b_1c_2| & |b_1d_2| & |b_1e_2| \\ & & |c_1d_2| & |c_1e_2| \\ & & & |d_1e_2| \end{array}\right\| \equiv 0$$

the five Pfaffians being got by deleting separately the five frame-lines of the quasi-Pfaffian array.

HELMIS, J. (1915): MACROBERT, T. M. (1916¼)

[Combinaisons déterminantes. *L'Enseignement Math.*, xx. pp. 271–276.]

[A method of obtaining examples on the multiplication of determinants. *Math. Notes* (Edinburgh Math. Soc.) i. pp. 230–231.]

The first note here concerns the number of combinations each consisting of m elements taken from an m-by-n array, subject to the condition that in the forming of a combination one element and only one shall be taken from each row.

D being a determinant with one or more known factors, and Δ a determinant got from the adjugate of D by removing some of its factors, the examples referred to in the second note are of the type $D \cdot \Delta$:

$$\text{e.g.}\quad \begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix} \cdot \begin{vmatrix} a+b & -a & -a \\ -a & a+b & -a \\ -a & -a & a+b \end{vmatrix}.$$

MUIR, T. ($1916^{30}/_5$)

[A sixth list of writings on determinants. *Quarterly Journ. of Math.*, xlvii. pp. 344–384.]

This list contains in all 374 titles, 156 belonging to the periods of the five preceding lists and 218 to the new five-year period 1911–1915. In the introduction to the list much satisfaction is expressed at the important step taken by the Mathematical Association in publishing * W. J. Greenstreet's Catalogue of Current Mathematical Journals, . . . with the names of the Libraries in which they may be found.

VOLTERRA, V. ($1917/_1$)

[The generalization of analytic functions. *Rice Institute Pamphlet*, iv. pp. 53–101.]

Near the beginning of his first lecture on this subject the author devotes a couple of pages (pp. 57–58) to what he calls 'General formulæ about matrices'. The single result arrived at, however, is a vanishing aggregate of products of pairs of determinants, the one factor of each product being of the r^{th} order and the other of the p^{th}. It is essentially identical with a theorem of Schweins' dating back to 1825 (*Hist.*, i. p. 171). The case where the orders are the same Volterra attributes to Antonelli, but it is really the so-called Sylvester theorem of 1839 (*Hist.*, i. p. 233).†

MUIR, T. ($1917^{28}/_3$)

[Note on an expansion of the product of two oblong arrays. *Transac. R. Soc. S. Africa*, vii. pp. 15–17.]

The new expansion takes the form of an aggregate of single determinants, the general theorem being: *The product of two* m-*by*-n *arrays* A, B *is expressible as an aggregate of single deter-*

* London, G. Bell and Sons; 40 pp.; 1913.

† In our account of Antonelli's paper (*Hist.*, iv. pp. 18–19) the suffixes of the k's in the more general theorem should be $t, 2, 3, \ldots, m$.

minants, the first of which is the product of the first k *columns of the arrays, and the others are formed from this by bordering; namely, bordering first in every way with one of the remaining columns from* A *and the corresponding column from* B; *secondly, with two of the remaining columns from* A *and the corresponding two from* B; *and so on, those having an odd number of lines in the border being negative and the others positive.* For example, when $n, m, k = 5, 3, 3$, we have

$$\left\|\begin{matrix} a_1 & a_2 & \cdots & a_5 \\ b_1 & b_2 & \cdots & b_5 \\ c_1 & c_2 & \cdots & c_5 \end{matrix}\right\| \cdot \left\|\begin{matrix} \alpha_1 & \alpha_2 & \cdots & \alpha_5 \\ \beta_1 & \beta_2 & \cdots & \beta_5 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_5 \end{matrix}\right\|$$

$$= \begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix} - \begin{vmatrix} P_1 & P_2 & P_3 & a_4 \\ Q_1 & Q_2 & Q_3 & b_4 \\ R_1 & R_2 & R_3 & c_4 \\ \alpha_4 & \beta_4 & \gamma_4 & \cdot \end{vmatrix} - \begin{vmatrix} P_1 & P_2 & P_3 & a_5 \\ Q_1 & Q_2 & Q_3 & b_5 \\ R_1 & R_2 & R_3 & c_5 \\ \alpha_5 & \beta_5 & \gamma_5 & \cdot \end{vmatrix} + \begin{vmatrix} P_1 & P_2 & P_3 & a_4 & a_5 \\ Q_1 & Q_2 & Q_3 & b_4 & b_5 \\ R_1 & R_2 & R_3 & c_4 & c_5 \\ \alpha_4 & \beta_4 & \gamma_4 & \cdot & \cdot \\ \alpha_5 & \beta_5 & \gamma_5 & \cdot & \cdot \end{vmatrix},$$

where $|P_1Q_2R_3|$ is the product of $|a_1b_2c_3|$ and $|\alpha_1\beta_2\gamma_3|$. The connection of the new expression with others of earlier date is pointed out (*Hist.*, ii. pp. 57, 199–200: iv. p. 64).

FROBENIUS, G. (1917[12]/4)

[Ueber zerlegbar Determinanten. *Sitzungsb.* . . . *Akad. der Wiss.* (Berlin), 1917, pp. 274–277.]

The first theorem proved here is a case of Muir's of 1872, the latter being: *If* k *elements of one row of a determinant of the* n*th* *order contain a common factor, which is also contained in the corresponding elements of* n — k *other rows, this factor is a factor of every term of the determinant.* With Frobenius the common factor is zero.* The second theorem proved is the converse of Frobenius' case: *If all the terms of an* n*-line determinant vanish, there must be for some value of* k *an* (n — k + 1)*-by-*k *array having nothing but zero elements.* With these two as auxiliaries he then re-establishes a theorem first enunciated by him in 1912, namely:

* Muir's theorem and proof were not referred to by us under the year 1872 (*Hist.*, iii. p. 43), preference being there given to a theorem of Whitworth's which from one point of view was more general.

If the elements of an n-*line determinant be independent variables, and certain of them be replaced by zeros without causing the determinant to vanish, then the determinant remains an irreducible function save when all the elements of an* n-*by*-(n — k) *array are zeros.* Lastly as a deduction from this he arrives at D. König's first theorem of 1915 (see above, p. 78), expressing in passing a not very hopeful opinion of the method of 'graphs' as an aid in the study of determinants.

CIPOLLA, M. (1917)

[Sulla sviluppo di un determinante secondo i minori di due matrici complementari. *Atti . . . Accad. Gioenia . . .* (Catania) (5) x. No. 19, 8 pp.]

This paper follows up Nicoletti's of 1902, which however he speaks of as giving an expansion of a determinant according to products of minors of the last p rows by minors of the last q columns. The value of the paper lies in the extreme care taken to produce an irreproachable proof.*

METZLER, W. H. ($1917^1/_6$)

[Vanishing aggregates. *Proceed. R. Soc. Edinburgh*, xxxvii. pp. 324–326.]

This is a following-up of the second part of Muir's paper of 1888 (*Hist.*, iv. pp. 38–40). The aggregates in question are aggregates of determinants formed from three or more given determinants in accordance with a definitely prescribed mode of transference of rows and columns, the result arrived at being an extension of Deruyts' theorem of 1882 where the number of given determinants is only two (*Hist.*, iv. pp. 15–16, 38–40). For example, if the given determinants be

$$|\,a_{11}\,a_{22}\,a_{33}\,a_{44}\,|, \quad |\,b_{11}\,b_{22}\,b_{33}\,b_{44}\,|, \quad |\,c_{11}\,c_{22}\,c_{33}\,c_{44}\,|,$$

and each determinant of the first aggregate is to contain two

* Like Nicoletti he refers to Hesse's paper of 1868: this reference now, however, is attached by him to a theorem of Cauchy's of 1812 (*Hist.*, iii. pp. 28–29: i. p. 105).

rows of a's, one row of b's, and one row of c's; and each determinant of the second aggregate to be similarly formed but with columns taking the place of rows; then the two aggregates are equal, the number of determinants in each aggregate being $4!/2!1!1!$ The theorem rests, like its narrower predecessor, on Laplace's expansion-theorem.

ROSS, C. M. ($1917^1/_{12}$, $1918^1/_2$): RITT, J. F. ($1918^{23}/_2$)

[Question 18558. *Math. Quest. and Sol.*, iv. Part 6.]

[Proof of the multiplication-formula for determinants by means of linear differential equations. *Bull. American Math. Soc.*, xxiv. p. 370.]

Ross's first result is Schläfli's of 1851 (*Hist.*, ii. pp. 52–53). The proof referred to was brought forward by its author as being curious rather than important. And Ross's second result in its final form is that if $|234|$, the last 3-line minor of the array

$$\left\| \begin{matrix} x_1 & a_1 & b_1 & c_1 \\ x_2 & a_2 & b_2 & c_2 \\ x_3 & a_3 & b_3 & c_3 \end{matrix} \right\|,$$

vanish, then

$$|123| \cdot |142| = \rho\, |134|^2,$$

where $\rho = B_1C_1/A_1{}^2$.

CULLIS, C. E. ($1918^3/_3$)

[On a special matrix of order six. *Bull. Calcutta Math. Soc.*, x. pp. 127–140.]

In the first section here (pp. 127–130), which is all that strictly concerns our subject, the well-known Schläfli determinant

$$|\, a_1{}^2b_2{}^2c_3{}^2 \quad b_2c_3 + b_3c_2 \quad c_3a_1 + c_1a_3 \quad a_1b_2 + a_2b_1 \,|$$

is once more shown to be equal to $|\, a_1b_2c_3 \,|^4$ (*Hist.*, ii. pp. 52–53), and to be of non-zero rank

$$6,\ 3,\ 1,\ 0$$

according as $|a_1b_2c_3|$ is of non-zero rank

$$3,\ 2,\ 1,\ 0.$$

The proof given of the latter result, although purely determinantal, does not supply the want which we referred to when speaking of Kürschák's more general theorem of 1900 (see above, p. 7). The nature of it will be understood when we mention one of the auxiliary facts made use of in it, namely, that the 3-by-6 array

$$\left\|\begin{array}{cccccc} a_1{}^2 & a_2{}^2 & a_3{}^2 & a_2a_3 & a_3a_1 & a_1a_2 \\ 2c_1a_1 & 2c_2a_2 & 2c_3a_3 & c_2a_3+c_3a_2 & c_3a_1+c_1a_3 & c_1a_2+c_2a_1 \\ 2a_1b_1 & 2a_2b_2 & 2a_3b_3 & a_2b_3+a_3b_2 & a_3b_1+a_1b_3 & a_1b_2+a_2b_1 \end{array}\right\|$$

has $|a_1b_2c_3|$ for a factor, the cofactor in three of the twenty cases being 0.

MUIR, T. (1918⁴/₄)

[The quadratic relations between the determinants of a 4-by-8 array. *Proceed. R. Soc. Edinburgh*, xxxviii. pp. 219–225.]

The matter here discussed is the theorem which expresses *the product of two determinants as a sum of like products*, and the main contribution made is a peculiarly compact and convenient notation for the products in question. The first auxiliary is the theorem that *the full number of binary products of the determinants of an* n-*by*-2n *array is*

$$\tfrac{1}{2}\{(C_{n,n})^2+(C_{n,n-1})^2+\ldots+(C_{n,0})^2\},$$

and that this is expressible as the sum of $\frac{1}{2}$(n + 1) *squares when* n *is odd and* $\frac{1}{2}$(n + 2) *squares when* n *is even.* For example, when n is 4 the number of products is 35, a number which is partitionable into the four squares 1^2, 2^2, 3^2, 4^2. The second help arises from numbering the columns of the array by

$$1,\ 2,\ 3,\ \ldots,\ n,\ 1',\ 2',\ 3',\ \ldots,\ n'$$

and from specifying the determinants by the numbers of the columns which compose them; for example, the determinants in the products

$$1234\cdot 1'2'3'4',\quad 123'4'\cdot 341'2',\ \ldots$$

Lastly, there is the condensation of this notation got by arranging the products as elements of determinants—an operation to which the products lend themselves with striking naturalness—and then denoting the said determinants by

$$|\, a_{rs} \,|, \quad |\, \beta_{rs} \,|, \; . \; . \; . \; ;$$

for example, when n is 4, we have

a one-line determinant $|\, 1234 \cdot 1'2'3'4' \,|$ denoted by a_{11},

a 3-line determinant

$$\left|\begin{matrix} 121'2' \cdot 3'4'34 & 121'3' \cdot 2'4'34 & 121'4' \cdot 2'3'34 \\ 131'2' \cdot 3'4'24 & 131'3' \cdot 2'4'24 & 131'4' \cdot 2'3'24 \\ 141'2' \cdot 3'4'23 & 141'3' \cdot 2'4'23 & 141'4' \cdot 2'3'23 \end{matrix}\right| \text{ by } |\, \beta_{11}\beta_{22}\beta_{33} \,|,$$

a 3-line determinant

$$|\, 123'4' \cdot 1'2'34 \quad 132'4' \cdot 1'3'24 \quad 142'3' \cdot 1'4'23 \,| \text{ by } |\, \gamma_{11}\gamma_{22}\gamma_{33} \,|,$$

and a 4-line determinant

$$|\, 12'3'4' \cdot 1'234 \quad 21'3'4' \cdot 2'134 \quad 31'2'4' \cdot 3'124 \quad 41'2'3' \cdot 4'123$$

by

$$|\, \delta_{11}\delta_{22}\delta_{33}\delta_{44} \,|.$$

The remainder of the paper (§§ 11–18) is occupied with application of the notation in the evolution and statement of many linear relations among the products; for example, *the single element* a_{11} *is equal to the sum of the elements in any row or column of* $|\, \delta_{11}\delta_{22}\delta_{33}\delta_{44} \,|$, *the sum of all the elements of the four determinants is* $8a_{11}$, *and so on.*

MILLER, G. A. (1918[13]/[4]): MÉTROD, G. (1919[1]/[8])

[Determinant groups. *Bull. American Math. Soc.*, xxv. pp. 69–75.]

[Question 4952. *L'Intermédiaire des Math.*, xxvi. p. 100.]

Naturally it is the theory of Groups rather than the theory of Determinants that profits here: in this respect a contrast will be found in Bagnera's paper of 1887 (*Hist.*, iv. pp. 33–34).

The question concerns inverted-pairs: it elicited nothing.

MUIR, T. (1918$^{10}/_{6}$)

[Note on the representation of the expansion of a bordered determinant. *Messenger of Math.*, xlviii. pp. 23–32.]

As a deduction from Arnaldi's main theorem of 1896 (*Hist.*, iv. pp. 432–433) and the main theorem of the paper now reached it is shown that *a determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary.* It is overlooked, however, that the said deduction is only a case of Nicoletti's theorem of 1902 (see above). Three useful facts in regard to the expansion are noted: (1) that if the orders of the two complementary arrays, P and Q say, be p and q, the orders of the minors in the four-factor products are

$$\begin{array}{cccc} p, & q, & 0, & 0 \\ p-1, & q-1, & 1, & 1 \\ p-2, & q-2, & 2, & 2 \\ \cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot \end{array}$$

(2) that the number of such products is

$$1 + (C_{p,1})^2(C_{q,1})^2 + (C_{p,2})^2(C_{q,2})^2 + \ldots,$$

and (3) that to obtain a specimen product it is well to fix first on the minors to be drawn from P and Q, next to delete from the entire determinant the rows and columns to which the said minors belong, and then in the minor resulting from this deletion to change all the P and Q minors into zeros.

MUIR, T. (1919$^{9}/_{3}$)

[Note on certain determinant identities arrived at by H. v. Koch. *Trans. R. Soc. S. Africa*, viii. pp. 101–105.]

The identities in question are those established in § 2 of Koch's first paper of 1890 on the solution of an infinite set of linear equations. After careful examination the main identity is shown to be practically the same as a result given by Schweins in 1825 concerning the quotient of two determinants. The character of the others will readily be guessed on a perusal of the account

given by us of Koch's paper at its proper historical place (*Hist.*, iv. pp. 418–419).

STRAZZERI, V. ($1919^{27}/_4$)

[Sullo sviluppo dei determinanti. *Rendic. del Circolo Mat.* (Palermo), xliii. pp. 352–356.]

The subject here is really Chio's condensation-theorem of 1853 and deductions from it (*Hist.*, ii. pp. 79–81: iii. pp. 80–81). The most important of the latter we may put concisely as follows: *If* D_n *be any* n*-line determinant, then the compound determinant whose elements are the* k*-line minors of* D_n *that each include the coaxial minor* D_{k-1} *is equal to* $(D_{k-1})^{n-k} \cdot D_n$. The fact however must not be overlooked that this is merely an instance of the class of theorems which we have spoken of as being ‘ extensionals of manifest identities ’ (*Hist.*, ii. pp. 58–61).

RICE, L. H. ($1920/_{10}$)

[On determinant expansions. *American Journ. of Math.*, xlii. pp. 237–242.]

Misled by Muir's paper of 1918 (see above, p. 86) Rice takes up the theorem which his predecessor ought to have attributed to Nicoletti, establishes it in his own way, gives repeated extensions to it, and brings it into useful relation with Albeggiani's theorem of 1874 (*Hist.*, iii. pp. 52–53). In the extensions he goes beyond Nicoletti. The latter, it may be remembered, restricted himself to arrays marked off by full-length horizontal and vertical lines, whereas with Rice the determinant is partitioned into rectangular arrays in any way.

In addition to thus throwing fresh light on the subject as confined to ordinary determinants, he then goes beyond our domain and lays down the analogous theorems in regard to p-dimensional or ‘ p-way ’ determinants.

LIST OF AUXILIARY WRITINGS

Mainly Applicational

1894. Grassmann, H. Gesammelte mathematische und physikalische Werke. Band I enthaltend die beiden Ausdehnungslehren und . . . xv + 435 pp. Leipzig.

1895. Bie, L. H. Kombinationslære. *Nyt Tidsskrift f. Mat.*, A., vi. pp. 81–126.

1905. Privorsky, A. A több változós függvények elméletéhez. *Math. és Phys. Lapok*, xiv. pp. 201–211.

1912. Padoa, A. Une question de maximum ou de minimum. *Proceed. Internat. Congr. Math.* (Cambridge), i. pp. 337–340.

1912. Frobenius, G. Ueber Matrizen aus nicht-negativen Elementen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1912, pp. 456–477.

1918. Müller, E. Beiträge zur Grassmannsche Ausdehnungslehre. *Sitzungsb. . . . Akad. d. Wiss.* (Wien) Abth. ii[a] cxxvii. pp. 1645–1699.

The following is the list referred to on page 76 above:—

1849. Cayley, A. On the order of certain systems of algebraical equations. *Camb. and Dub. Math. Journ.*, iv. pp. 132–137.

1855. Salmon, G. On the order of certain systems of equations. *Quart. Journ. of Math.*, i. pp. 246–257.

1866. Salmon, G. Modern Higher Algebra. Chap. XVIII, pp. 212–238.

1867. Roberts, S. Sur l'ordre des conditions de la coexistence des équations algébriques à plusieurs variables. *Crelle's Journ.*, lxvii. pp. 266–278.

1872. Brill, A. Ueber Elimination aus einem gewissen System von Gleichungen. *Math. Annalen*, v. pp. 378–396.

1875. Roberts, S. On a simplified method of obtaining the order of algebraical equations. *Proceed. London Math. Soc.*, vi. pp. 101–113.

1876. Salmon, G. Modern Higher Algebra. Chap. XVIII, pp. 238–266.

1885. SALMON, G. Modern Higher Algebra. Chap. XIX, pp. 283–313.

1890. BRILL, A. Ueber algebraische Correspondenzen. *Math. Annalen*, xxxvi. pp. 321–360.

1900. SEGRE, C. Gli ordini delle varietà che annullano i determinanti dei diversi gradi estratti da una data matrici. *Atti . . . Accad. dei Lincei, Rendic.* (5) ix. pp. 253–260.

1902. PALATINI, F. L'ordine della varietà che annulla i subdeterminanti di un dato grado di un determinante emisimmetrico. *Atti . . . Accad. dei Lincei, Rendic.* (5) xi. pp. 315–318.

1903. GIAMBELLI, G. Z. Ordine della varietà rappresentata coll' annullare tutti i minori . . . *Atti . . . Accad. dei Lincei, Rendic.* (5) xii. pp. 294–297.

1904. GIAMBELLI, G. Z. Ordine di una varietà più ampia di quella . . . *Mem. . . . Ist. Lombardo*, xx. pp. 101–103.

1905. GIAMBELLI, G. Z. Sulle varietà . . . *Atti . . . Accad. . . . di Torino*, xli. pp. 102–105.

1909. OEHLER, H. Ueber die Gleichungssysteme, welche man aus einer Matrix variabler Elemente durch Nullsetzen der Determinanten gegebener Ordnung erhält. Dissert. 58 pp. Tübingen.

1918. DECKER, F. F. On the order of a restricted system of equations. *American Journ. of Math.*, xli. pp. 283–298.

CHAPTER I(a)

HADAMARD'S APPROXIMATION-THEOREM, FROM 1900 TO 1917

There are two rather important theorems in determinants which both bear Hadamard's name, one belonging to the year 1893 and the other to 1903. As confusion has arisen by each of them being called simply 'Hadamard's theorem', I propose, as in the above title, to call the earlier of them his *approximation*-theorem and the later his *evanescence*-theorem. In doing so, however, I am also in duty bound to recall the fact that the former theorem has a claim to bear also Lord Kelvin's name, he having been occupied with it eight years in advance of Hadamard,* and in connection with the latter to draw attention to the earlier results of Lévy (1881) and Desplanques (1886).

PETRINI, H. ($1901^{30}/_{3}$)

[Nota sobre la transformación ortogonal de una determinante. *Revista trim. de Mat.*, i. pp. 11–15.]

A start is here made with three determinants Δ, D, δ, all of the n^{th} order, and all with complex elements, the determinants got from them by changing every element into its conjugate complex being denoted by $\overline{\Delta}$, $\overline{D}$, $\overline{\delta}$ respectively. Then the supposition being made that

$$\Delta = \delta \overset{c}{\underset{c}{\times}} D \quad \text{and} \quad \delta \overset{r}{\underset{r}{\times}} \overline{\delta} = 1,$$

it is affirmed that not only is

$$\Delta\overline{\Delta} = D\overline{D},$$

* For a short statement of the related facts in regard to this, see *Transac. R. Soc. S. Africa*, i. p. 323, footnote.

but that every diagonal element of $\Delta\bar{\Delta}$ is equal to the corresponding element of $D\bar{D}$. Next are brought forward two other results asserted to rest on the same hypothesis, namely,

$$D = \bar{\delta} \overset{r}{\underset{c}{\times}} \Delta \quad \text{and} \quad \delta \overset{c}{\underset{c}{\times}} \bar{\delta} = 1;$$

in this case, however, proof is not considered to be negligible. In the second section the further supposition is made that the elements of δ can be so chosen as to make all the terms of Δ vanish except the diagonal term, this leading later to the important conclusion

$$\Delta\bar{\Delta} \leqq \text{its diagonal term},$$

the well-known equivalent of Hadamard's approximation-theorem of 1893 (*Hist.*, iv. pp. 483–484).

NANSON, E. J. (1901/$_7$)

[A determinant inequality. *Messenger of Math.*, xxxi. pp. 48–50.]

The inequality referred to is that communicated by Sir William Thomson (afterwards Lord Kelvin) to Muir in 1885 (*Hist.*, iv. p. 32), namely, that *any determinant* $< \sigma_1\sigma_2 \ldots \sigma_n$ *where* σ_r *is the square root of the sum of the squares of the elements of its* rth *row.* Here it is first based on geometrical considerations, which in the case of the 3rd order are (*a*) that if $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$ be the other extremities of three edges of a parallelopiped that meet in the origin, the volume of the parallelopiped (*Hist.*, i. p. 348) is $|\, x_1y_2z_3 \,|$; (*b*) that a parallelopiped whose edges are fixed in length is of greatest volume when the edges are mutually perpendicular. Next, the theorem is shown to be easily deducible from a property of a special axisymmetric determinant, namely, that *any positive axisymmetric determinant whose coaxial minors of every order are positive can never be greater than the product of its leading diagonal elements.* The deduction is possible because the square of any determinant is known to be such a special axisymmetric, having in fact any of its r-line coaxial minors expressible as the sum of $(n)_r$ squares. The author also does not fail to note that the square of an oblong array is such another, and he formulates the wider derived result

FISCHER, E. ($1907^1/_6$, $1908/_1$)

[Ueber den Hadamardschen Determinantensatz. *Archiv d. Math. u. Phys.*, (3), xiii. pp. 32–40.]

Much of this is occupied with the consideration of Hermitian forms—that is to say, with quadrics of the type $\Sigma a_{ik}x_i\bar{x}_k$, where the a's and x's are complex and $\bar{x}_k$ is conjugate to x_k. Indeed, it appears to be considered that for the purposes of exposition a combination of the two subjects, the form and its determinant, is preferable; and half the paper is taken up in familiarizing the reader with the inter-relations of the two, approved restatements of Hadamard's theorem and lemma being given with this end in view. The course thus taken is so far justified in that it leads to the establishment of an interesting generalization of the said lemma, namely, *The determinant of every positive definite Hermitian form is not greater than the product of any coaxial minor and its complementary.* As an example of such a form, to which this and Hadamard's theorem are usefully applicable, there is brought forward the case where the coefficients a_{ik} are of the form

$$\int_a^b u_i(s) \cdot u_k(s)ds,$$

and where the u's are real or complex functions, continuous between the limits a and b, and the variable s is real and continuous. Lastly, connected with this determinant of definite integrals, we not unexpectedly come on the condition that the u's as above specified will be linearly independent if

$$\begin{vmatrix} \int_a^b u_1{}^2 ds & \int_a^b u_1u_2 ds & \dots & \int_a^b u_1u_n ds \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \int_a^b u_nu_1 ds & \int_a^b u_nu_2 ds & \dots & \int_a^b u_n{}^2 ds \end{vmatrix} \lessgtr 0,$$

and we are thus brought into touch with a rival of the Wronskian.

WIRTINGER, W. (1907/$_7$)

[Zum Hadamardschen Determinantensatz. *Monatshefte f. Math. u. Phys.*, xviii. pp. 158–160; or, in French, *Bull. des Sci. Math.*, (2), xxxi. pp. 175–179.]

What Wirtinger gives us is a quite fresh proof. His mode of procedure is a direct application of the ordinary Lagrangian rule for finding by differentiation the extreme values of a function whose variables are connected by equations of condition. His remarks on the corresponding geometrical theorem coincide with those of Nanson in his paper of 1901 (see above, p. 91).

SHARPE, F. R. (1907/$_{10}$)

[The maximum value of a determinant. *Bull. American Math. Soc.*, xiv. pp. 121–123.]

The subject here is those maximum determinants whose elements are either 1 or -1. Taking such a determinant of order $4p$ and value $(4p)^{2p}$ the author finds the values of its minors

$$\begin{array}{lll} \text{of order } 4p-1 & \text{to be} & (4p)^{2p-1} \text{ or } -(4p)^{2p-1}, \\ \quad \ldots \quad 4p-2 & \ldots & 2(4p)^{2p-2} \text{ or } -2(4p)^{2p-2} \text{ or } 0, \\ \quad \ldots \quad 4p-3 & \ldots & 4(4p)^{2p-2} \text{ or } -4(4p)^{2p-2} \text{ or } 0. \end{array}$$

Also by bordering it with the permissible elements he constructs a determinant

$$\text{of order } 4p+1 \text{ whose value is } 4(4p-1)\,(4p)^{2p-1}.$$

By this means it is thought that possibly maximum determinants whose order is not a multiple of 4 may have been lit upon. In any case it is important to note that Davis' theorem of 1882 on the same subject (*Hist.*, iv. p. 18) is shown to be quite unreliable. According to the latter the maximum determinant of the 7^{th} order is the circulant $C(-1, 1, 1, 1, 1, 1, 1)$, whose value is 320; whereas, as we now learn, a rival is ready to hand whose value is 512, namely, the cofactor of the $(1, 1)^{\text{th}}$ element in Hadamard's 8-line maximum determinant of 1893 (*Hist.*, v. pp. 483–484).

MUIR, T. ($1908^{28}/_{12}$)

[An upper limit for the value of a determinant. *Transac. R. Soc. S. Africa*, i. pp. 323–334.]

Unlike the preceding writings the present deals not only with Hadamard's theorem specially so called, but with all the other contents of his paper of 1893. Naturally at the outset the writer recalls Lord Kelvin's theorem, which preceded Hadamard's, and his own lemma by which Lord Kelvin's theorem was proved, namely, *If* s_r *be the square root of the sum of the squares of the elements of the* r^{th} *row of any determinant* D, *and* S_r *be the like square root in the case of the adjugate, then*

$$D \leqq s_r S_r.$$

He then not only extends his proof of 1885 (*Hist.*, iv. p. 32) to cover Hadamard's more general theorem, using for the purpose the multiplication of a pair of special 2-by-n arrays, but gives an alternative proof that is also purely determinantal, based on Chio's condensation-theorem. The conditions under which Hadamard's limit may be actually reached are next discussed, the circumstances under which a determinant may appropriately be spoken of as having a maximum value or of being a maximum determinant, and the cause of the close relation between such a determinant and the inversely orthogonal determinant (or ant-orthogonant) which Sylvester had studied in 1867 (*Hist.*, iii. pp. 478–480). The main facts so brought out may be condensed thus: *If* δ, δ' *be determinants whose corresponding elements are conjugate complex, and* Δ, Δ' *be their respective adjugates, then* (1) *when the elements of* δ *are proportional to the corresponding elements of* Δ' *the value of* δ *is a maximum*; (2) *when in addition the elements of* δ *are equimodular* δ *is inversely orthogonal*; *and* (3) *when* δ *is inversely orthogonal and has equimodular elements its value is a maximum*. In support of Sylvester it is then proved that *the alternant of the* n^{th} *order* $|\, x^0y^1z^2w^3 \ldots \,|$ *will be inversely orthogonal if* x, y, z, w, . . . *be the* n^{th} *roots of a constant*, and, as a preparation for Hadamard, that *whatever* d *may be, the determinant*

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & d & -d \\ 1 & -1 & -d & d \end{vmatrix}$$

is inversely orthogonal, the product of any element by its cofactor in the determinant being —4d, *and therefore the value of the determinant itself* —16d. Regarding the formation of Hadamard's maximum determinants with unit elements it is shown (1) that *if a solution be obtained for order* r *it is easy to give a solution for order* 2r; (2) that *those of order* 2^m *are axisymmetric*; and (3) that *the one of order* 12 *is axisymmetric, but not the one of order* 20. Finally, maximum determinants with real elements other than unity are considered; like those just spoken of they are shown to be orthogonants, and examples are given that are also skew.

KLUGE, W. (1908)

(See under this heading in Chapter XI.)

SCHUR, I. (1909 early)

[Ueber die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen. *Math. Annalen*, lxvi. pp. 488–510.]

In this paper Hadamard's theorem makes a merely incidental appearance (§ 5) as being included in a theorem on latent roots. The latter is in effect that *the sum of the norms of the latent roots of a determinant is not greater than the sum of the norms of the elements*; and the passage from it to Fredholm's case of Hadamard's theorem is easily effected by first using the familiar inequality connecting the arithmetical and geometrical means in order to bring in the product of the norms instead of the sum, and then by substituting the determinant itself for the said product.

HAYASHI, T. ($1909/_7$, $1910/_4$)

[Hadamard's theorem on the maximum value of a determinant. *Tôkyô Sûgaku-Butur. Kizi*, (2), v. pp. 104–109; also. in French, *Giornale di Mat.*, xlviii. pp. 253–258.]

The writer here establishes the special case of Hadamard's theorem where the elements are all real and the sum of the

squares of the elements of each row is 1. The mode of proof is similar to Muir's second mode as applied to the general theorem, being based on Chio's condensation result of 1853.

KOWALEWSKI, G. (1909)

[Einführung in die Determinantentheorie, . . . v. + 550 pp. Leipzig.]

Kowalewski confines himself (pp. 458–460) to the case of Hadamard's theorem where the elements are real; and his mode of proof is not essentially different from the original of 1893, thus merely turning upon the fact that *the determinant which is the square of an* m-*by*-n *array of real elements is less than the product of its* (m, m)th *element by the cofactor of that element.*

TONELLI, L. (1909)

[Sul teorema di Hadamard relativo al valor maggiorante di un determinante. *Giornale di Mat.*, xlvii. pp. 212–218.]

The theorem in question is here put in the specialized form used by Fredholm and spoken of above as Fredholm's case, namely, *If the absolute value of every element of an* n-*line determinant be less than a positive quantity* β, *the absolute value of the determinant itself is not greater than* $\beta^n\sqrt{n^n}$; and the details of the proof are concerned with a still further specialization made originally by Hadamard himself, namely, *If in each row of a determinant the sum of the squares of the absolute values of the elements be* 1, *the determinant itself is in absolute value not greater than* 1. Tonelli's discussion of the theorem is full and thorough. The ultimate basis of it, as in the case of the first of Muir's proofs, is the theorem regarding the product of a pair of special 2-by-n arrays. A section is devoted (pp. 213–215) to the case where the elements are real: another section to the case where they are, as with Nanson, co-ordinates of points in n-dimensional space: and a third (pp. 216–218) to the case where they are complex.

AMOROSO, L. (1910/$_6$)

[Sul valore massimo di speciali determinanti. *Giornale di Mat.*, xlviii. pp. 305–315.)

The main part (§ 1) of this paper is occupied with the finding of upper limits for

$$\begin{vmatrix} \int_a^b f_1(x) \cdot u_1(x)d(x) & \ldots & \int_a^b f_1(x) \cdot u_n(x)dx \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \int_a^b f_n(x) \cdot u_1(x)dx & \ldots & \int_a^b f_n(x) \cdot u_n(x)dx \end{vmatrix}^2,$$

where the functions involved, both the f's and the u's, are real and integrable between the limits a and b. The remainder (§ 2) discusses cases where specialization is made in the functions, and ends by showing that Hadamard's theorem for real elements is one of these.

SZÁSZ, O. (1910, 1911, 1914)

[Az Hadamard-féle determinánstétel egy elemi bebizonyitása. *Math. és Phys. Lapok*, xix. pp. 221–227.]

[Ein elementarer Beweis des Hadamardschen Determinantensatzes. *Math. u. naturw. Berichte aus Ungarn*, xxvii. pp. 172–180.]

[Egy determinánstélelröl. *Math. és Phys. Lapok*, xxiii. pp. 1–4.]

Szász's mode of treating the theorem is quite different from any of the preceding. The Hermitian determinants concerned being $|a_{rs}|_n$ and $|a'_{rs}|_n$, where a_{rs} and a'_{rs} are conjugate complex, and their product $|p_{rs}|_n$, he seeks of course to prove that

$$|p_{rs}|_n \leqq p_{11}p_{22}\cdots p_{nn}.$$

To this end he sets about transforming

$$|a_{rs}|_n \text{ and } |a'_{rs}|_n \quad \text{into} \quad |b_{rs}|_n \text{ and } |b'_{rs}|_n$$

in such a way that the product

$$|b_{rs}|_n \cdot |b'_{rs}|_n, \quad \text{or} \quad |q_{rs}|_n \text{ say,}$$

may have its non-diagonal elements all zeros—in other words, so that

$$\left| q_{rs} \right|_n = q_{11}q_{22} \cdot \cdot \cdot q_{nn}.$$

His next task is to show that

$$p_{11} = q_{11}, \quad p_{22} \geqq q_{22}, \quad p_{33} \geqq q_{33}, \quad \cdot \cdot \cdot , \quad p_{nn} \geqq q_{nn}$$

which easily leads him to

$$p_{11}p_{22} \cdot \cdot \cdot p_{nn} \geqq q_{11}q_{22} \cdot \cdot \cdot q_{nn},$$

i.e. $\geqq \left| q_{rs} \right|_n$

i.e. $\geqq \left| p_{rs} \right|_n,$

as desired. It must be noted, however, that neither of the two portions of the proof, though both are elementary enough, are such as bring conviction in a line or two; and still greater fullness might have been advantageous. The transformation of $\left| a_{rs} \right|_n$ into $\left| b_{rs} \right|_n$ is effected by taking over the elements of the first row unchanged, by taking for the second row the original second row diminished by x_{21} times the new first row, for the third row the original third row diminished by x_{31} times the new first row and x_{32} times the new second row, and so on, the values of the x's being determinable by means of the conditions attaching to the non-diagonal elements of the product $\left| b_{rs} \right|_n \cdot \left| b'_{rs} \right|_n$. As a matter of fact the actual determination is not carried out, as it suffices in the second part of the proof to use the conditional equations instead.

Attention is next devoted to Fischer's generalization of 1907, and Szász gives a proof of it in the form: *If* M *be an oblong array of complex elements, then* MM′ *is not greater than the product of any two complementary coaxial minors of* MM′. Also, following up an idea of Nanson's, he shows that *If every coaxial minor of a Hermitian determinant be positive, then the determinant is not greater than the product of any pair of its complementary coaxial minors.*

BOGGIO, T. (1911/$_4$)

[Nouvelle démonstration du théorème de M. Hadamard sur les déterminants. *Bull. des Sci. Math.*, (2) xxxv. pp. 113–116.]

This clearly worded demonstration, though not identical with Szász', is essentially on the same lines. It might be called a new and improved version.

KNESER, A. (1911)

[Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. viii + 243 pp. Braunschweig.]

As might be expected in a text-book with the above title, the author devotes a full section (§ 61, pp. 227–231) to Hadamard's theorem and its cases; and, as he bases his proof on the theory of orthogonal transformation, he naturally opens with a page or two on this subject, giving a careful proof of the theorem that *An orthogonal substitution is always formable in which the coefficients of a row are proportional to given quantities that are not all zero.*

HEYWOOD, H. B., AND FRECHET, M. (1912)

[L'équation de Fredholm, et ses applications à la physique mathématique (chap. ii. pp. 50–52). vi + 165 pp. Paris.]

The proof of Hadamard's theorem here given is on the same lines as Wirtinger's of 1907, differentiation being used. The elements of the determinant are taken to be real.

KUBOTA, T. ($1912^1/_6$)

[Hadamard's theorem on the maximum value of a determinant. *Tôhoku Math. Journ.*, ii. pp. 37–38.]

The proof here given concerns the case where the elements are real. In style it is gradational, the n^{th} case being deduced from the $(n-1)^{\text{th}}$ by the use of an orthogonal substitution.

CIPOLLA, M. ($1912^{27}/_8$)

[Sul teorema di Hadamard relativo al modulo massimo di un determinante. *Giornale di Mat.*, l. pp. 355–359.]

Here the writer offers what he justly calls a direct and very simple proof founded on elementary properties of the product of two oblong arrays and on the development of a determinant according to products of the elements of a row and column; and what we actually get is naturally a simplified edition of Hadamard's own proof.

BLASCHKE, W. (1912/9)

[Ein Beweis für den Determinantensatz Hadamards. *Archiv d. Math. u. Phys.*, (3) xx. pp. 277–279.]

The proof in question is different from any of those preceding it, but the difference is really somewhat less than might at first sight appear. To commence with, the author says that by expanding $|a_{11} \ldots a_{nn}|$ in terms of the elements of its first row and their cofactors he obtains

$$\begin{aligned} &|a_{11} \ldots a_{nn}|\,|a'_{11} \ldots a'_{nn}| \\ &= \Sigma a_{1k}a'_{1k} \cdot \Sigma A_{1k}A'_{1k} - \Sigma(a_{1r}A'_{1s} - a_{1s}A'_{1r})(a'_{1r}A_{1s} - a'_{1s}A_{1r}) \\ &> \Sigma a_{1k}a'_{1k} \cdot \Sigma A_{1k}A'_{1k}, \end{aligned}$$

not noting the fact that he is actually using like Muir (1908) the theorem regarding the two equivalents for

$$\left\| \begin{matrix} a_{11} & a_{12} & \ldots & a_{1n} \\ A'_{11} & A'_{12} & \ldots & A'_{1n} \end{matrix} \right\| \quad \left\| \begin{matrix} a'_{11} & a'_{12} & \ldots & a'_{1n} \\ A_{11} & A_{12} & \ldots & A_{1n} \end{matrix} \right\|.$$

The after-procedure is made clearer, as in the case of other proofs, by the author pointing out the meaning of each corresponding step of it when $a_{r1}, a_{r2}, \ldots, a_{rn}$ are taken to be the co-ordinates of a vector in n-dimensional space.

DIXON, A. C. (1913⁹/4)

[On the greatest value of a determinant whose constituents are limited. *Proceed. Cambridge Philos. Soc.*, xvii. pp. 242–243.]

What is contained here is a fresh proof of Hadamard's theorem in the form enunciated by Szász, save that the start is now made from an oblong array instead of a square array. In character it is gradational, and might quite fairly have borne Szász' descriptive term 'elementary'. To establish the theorem for an m-by-n array use is twice made of it for an $(m-1)$-by-n array, namely, once in regard to a primary coaxial minor of the product and once in regard to a primary coaxial minor of the adjugate of the product.

MOLINARI, A. M. ($1913^1/_7$)

[Sul teorema di Hadamard. *Atti . . . Accad. dei Lincei (Rendiconti)*, (5) xxii. pp. 11–12.]

The form here established is the so-called Fredholm's. The proof is essentially geometrical, actual orthogonal transformation being utilized.

PETROVITCH, M. (1913, 1914)

[Theorem on the maximum modulus of a determinant, and several of its applications (In Serbian). *Rad jugoslav. Akad.* . . ., cc. i. 18 pp.]

[Quelques conséquences du théorème sur le maximum du déterminant. *Izvješća*, i. pp. 65–67.]

The theorem in question is derived from, and may be viewed as being really only a variant of, Hadamard's, the change in form being got by utilizing the inequality connecting the arithmetic and geometric means, as had already been done by Schur in 1909. We may concisely express it thus: *The norm of an* n-*line determinant having* N *for the sum of the norms of its elements is* $\{N/n\}^n$. One of the applications made of it is to find an upper limit for the radius of the bounding circumference of the solutions of a set of linear equations.

SZÁSZ, O. (1917 late)

[Ueber eine Verallgemeinerung des Hadamardschen Determinantensatz. *Monatshefte f. Math. u. Phys.*, xxviii. pp. 253–257.]

The main theorem here established, although rightly viewed as the title indicates, is more manifestly a generalization of Nanson's hitherto unproved theorem of 1902 regarding axisymmetric determinants. In fact what Szász proves is that the said theorem of Nanson holds not only for an axisymmetric determinant but also for a Hermitant whose conjugate elements are conjugate complex.

CHAPTER II

TEXTBOOKS FROM 1900 TO 1919

By far the most striking feature connected with this chapter is the great falling off in the number of writings chronicled as compared with the number belonging to the immediately preceding 20-year period. Instead of having well over eighty (80) we have now just under forty (40). Doubtless the fact that the Great War occupied a goodly part of the new period provides a sufficient explanation. It is not enough, however, merely to recall the war; detail must be given to explain why in almost every other chapter the writings dealt with show increases, and why the total number of writings, when we take into account all the chapters, is vastly greater for the war-stricken period than for its predecessor.

The number of languages represented is one or two fewer than formerly: German still heads the list, English being a poor second, and Italian a poor fourth. What is noteworthy is the change in the position of Russian, which now occupies the third place. Unfortunately for us such Russian manuals it has latterly been impossible to procure.

PASCAL, E. (1900)

[Die Determinanten. Eine Darstellung ihrer Theorie und Anwendungen mit Rücksicht auf die neuen Forschungen. Berichtigte deutsche Ausgabe von H. Leitzmann. xvi + 266 pp. Leipzig.]

A carefully produced translation. Additions to the text are noticeably few; on the other hand the historical notes have been considerably increased, and all the titles in them, new and old, are given in a full and workmanlike manner. One particularly noteworthy fact under this head is that Reiss' *Beiträge* is for the

first time referred to by any editor of a book on Determinants. As for corrections, very few have been ventured on, a fact all the more to be regretted because the popularity of the book has only served to spread more widely some serious inaccuracies as to the history of theorems.

PRANG, C. (1900): LELIEUVRE, M. (1901)

[Einführung in die Theorie und den Gebrauch der Determinanten. iv + 53 pp. Berlin.]

[Sur la théorie des déterminants. *L'Enseignement Math.*, iii. pp. 205–208.]

The first contribution here is a useful school textbook of a type already common, and having no outstanding feature of merit. About a dozen pages are devoted to analytic geometry.

The second is an attempt to give an account of the elementary theory in three pages.

SCHMEHL, CHR. (1901, 1913): GROAT, B. F. (1902): PRADO, G. F. DE (1902)

[Die Algebra und die algebraische Analysis, mit Einschluss . . ., viii + 286 pp. Giessen.]

[Seven lessons in theory of inversions of order and determinants. 32 pp. Minneapolis.]

[Elementos de la teoria de los determinantes. . . . 2ª edición. . . ., xiii + 324 pp. Madrid.]

The first of these is a manual of Algebra for use with the upper classes of higher-grade schools. It concludes with a chapter on determinants, which in the case of the second edition extends to forty-one pages.

The second item may with accuracy be described, like so many booklets that have passed before us, as an 'introduction to the elementary theory of determinants': and in the qualities of clearness and general helpfulness it is above the average of such.

The third has already received attention (*Hist.*, iv. p. 91).

CAPELLI, A. (1902, 1909)

[Istituzioni di Analisi Algebrica. 3ª ed. con aggiunte delle "Lezioni di algebra complementare". xix + 714 pp. 4ª ed. notevolmente ampliata. xxviii + 953 pp. Napoli.]

Attention is duly given here to determinants, but special interest attaches to the full and fresh treatment of linear equations, a treatment, however, fairly to be expected from what we already know of the author (*Hist.*, iv. pp. 102–103, 104–105).

KRONECKER, L. ($1903^{28}/_{7}$)

[Vorlesungen über die Theorie der Determinanten. Erster Band. Bearbeitet und fortgeführt von K. Hensel. xii + 390 pp. Leipzig.]

The lectures in question, twenty-one in number, are part of a university course on Allgemeine Arithmetik belonging to the period 1883–1891.

Apart from the name of the lecturer interest is at once awakened in the work by the fact that there are 390 pages in what is announced to be only the first volume of it. On looking inside it is soon seen that as many as five lectures (II–VI) are concerned with determinants of the second order, five lectures (VII–XI) with determinants of the third order, and that the n-line determinant is not met with until we reach the sixteenth lecture and the 263rd page. A little closer examination shows that the term 'rank' in its special sense as connected with non-evanescence is made use of near the very outset in treating of 2-line determinants, while the definition of a determinant engages attention in the seventeenth and eighteenth lectures, and the term 'adjugate' as naming an important derived form of determinant does not make its appearance until the nineteenth lecture. Much space, we also soon learn, is given to linear equations and to geometrical applications, even a full lecture being occasionally assigned to each of them: an entire lecture, too, is allotted to matrices, another to invariant factors, and a third, in which determinants are never mentioned, to domains of rationality.

Of course the new features of the volume are not confined to its plan or to the choice and arrangement of material: the actual

treatment of the individual subjects is almost equally refreshing. No student who reads it through with care can fail to have his view widened and see his subject in truer perspective than before.

KÖNIG, J. (1903): DÖLP, H. (1903)

[Einleitung in die allgemeine Theorie der algebraischen Gröszen. xii + 564 pp. Leipzig. Also, in Magyar, xii + 599 pp. Budapest.]

[Die Determinanten . . . 6 Aufl. iv + 95 pp. Darmstadt.]

Determinants as a subject for exposition do not find a place in this book of König's, but in the treatment of other subjects they are freely made use of, and notably certain special forms of them. Thus, under symmetric functions alternants are employed, and under elimination bigradients and Jacobians. To Jacobians, indeed, unique consideration is given; the equivalent of a small chapter (pp. 243–259) is devoted to an exposition of certain of their properties. Any contribution made in the book to the theory of such special forms we shall deal with in their proper places.

The sixth edition of Dölp's booklet is simply a reprint.

GRACE, J. H. AND YOUNG, A. (1903)

[The Algebra of Invariants. vii + 384 pp. Cambridge.]

The Algebra of Invariants, as treated of here, is in particular and in the main that originated by Clebsch and Gordan, and so effectively used by them and their co-workers. So far as determinants are concerned the treatise is of course chiefly applicational; but the properties utilized have also a certain additional interest for the student of the theory, because of the fresh form in which they are presented to the eye.

SCOTT, R. F. (1904/$_5$)

[The Theory of Determinants and their applications. Second Edition, revised by G. B. Mathews. xii + 288 pp. Cambridge.]

The amount of revision made is considerable. In the first

place, three new chapters have been added, namely, an Introductory chapter (pp. 1–7), a chapter on Invariant Factors (pp. 75–87), and a chapter on Determinants of Infinite Order (pp. 120–129). In the next place, three of the old chapters have been more or less added to and altered, these being Chap. VI (pp. 60–79) on Compound Determinants, Chap. XI (pp. 130–148) on the Theory of Equations, and Chap. XIV (pp. 180–202) on Bilinear and Quadratic Forms. At the close there has also been inserted for the first time a sketch of the History (pp. 286–288). In this and other ways the revision constitutes a distinct improvement. Nor is the enlargement so great as might be expected, for the bibliographical list which formerly occupied ten pages is now omitted, professedly because of the accessibility of Muir's lists in the Quarterly Journal of Mathematics.

BAUER, G. (1903/$_7$): ORLANDO, L. (1903)

[Vorlesungen über Algebra. vi + 376 pp. Zweite Auflage. vi + 366 pp. 1909. Leipzig.]

[Exercizi d' Algebra svolti nell' Università di Messina. I. Determinanti e Sostituzioni lineari. (Lithogr.) 40 pp. Messina.]

The fourth section of these Lectures bears the title ' Theory and Application of Determinants ', and occupies about a quarter of the book (pp. 257–350). It is subdivided into seven chapters, two of which extending to a little over 30 pages are devoted to the pure theory. As might be expected all the applications are algebraical.

About half of Orlando's twenty-three so-called exercises are known properties of determinants, accompanied by proofs or comments. The principle of selection is not apparent.

GARBIERI, G. (1904)

[Teoria dei determinanti. Riassunto di lezioni date nelle università di Padova e Genova. 32 pp. Torino.]

Although this is said to be merely a ' synopsis of lectures ', it forms for its size quite a good introductory textbook of the subject.

NETTO, E. AND VOGT, H. (1904$^{12}/_{8}$)

[Déterminants. *Encyl. des Sci. Math.*, I(i), pp. 88–132.]

Great changes have been made on the original German edition (*Hist.*, iv. p. 95), one evidence being that there are now forty-five (45) pages of matter instead of eleven (11), and that the number of historical footnotes has grown from 66 to 179. Corrections have also been made, but more sparingly, and a few new inaccuracies have crept in. To a number of the latter attention is drawn in the brown-paper leaflets called 'Tribune publique' issued along with the various fascicles of the work; but more of such corrections are needed. Greatly improved though the article thus is, a searching revise would make it exceptionally useful.

The preceding article on Combinatory Analysis (pp. 63–68) has been similarly enlarged and improved.

ZAHRADNIK, K. (1905): PARFENTIEFF, N. N. (1906)

[O determinantech. iv + 51 pp. V Brně.]

[Theory of determinants (In Russian). 109 pp. Kasan.]

A useful elementary textbook for schools, but showing less improvement than might fairly be expected of a work published 35 years after Studnička's similar booklet with the same title.

The second is of a higher grade, consisting of lectures delivered in the University of Kasan and prepared for the use of students by St. Sowlew.

BES, K. (1905–1910):

VILLAFAÑE Y VIÑALS, J. M. (. . . 1906)

[Uit de theorie der algebraische vergelijkingen. *Wiskundig Tijdschrift*, ii. pp. 2–16, 49–57, 195–224; iii. pp. 65–81; iv. pp. 161–181; v. pp. 161–176; vi. pp. 101–117, 237–253; vii. pp. 13–24. Also separately, 160 pp. Haarlem.]

[Elementos de la teoria Coordinatoria y de las Determinantes. 5ª edición. . . . pp. Barcelona.]

The title of Bes' interesting piece of work agrees with its contents, the portions of the theory that are dealt with being those usually requiring determinants for their exposition. The first of the twelve chapters is thus, not unnaturally, devoted to determinants pure and simple, and the third to the minor determinants of an oblong array. It must be added that the book is not commonplace in execution, and that few textbooks on the Theory of Equations would be found to treat so fully of non-linear equations of more than one unknown. A few suitable exercises are appended to each chapter.

The other textbook has not been seen in any of its editions.

WELD, L. G. ($1906^1/_1$): CARDOSO-LAYNES, G. (1906)

[Determinants. 4th edition, enlarged. 56 pp. New York.]

[Una lezione su la teoria elementare dei determinanti. *Supplemento al Periodico di Mat.*, x. pp. 67–76.]

The first of these we have only hitherto seen as a portion of a textbook on higher mathematics (*Hist.*, iv. pp. 93–94). With its appearance as a separate publication short articles are given on the applications of determinants to higher algebra. The only special forms considered are Jacobians and Hessians.

The second is an introductory exposition of the elementary properties: it is not so futile as the space occupied might suggest.

BÖCHER, M. (1907)

[Introduction to Higher Algebra. xi + 321 pp. New York.]

The title here cannot fail to call to mind Salmon's 'Lessons Introductory to the Modern Higher Algebra', and it is well, therefore, at once to note that the same closeness of resemblance does not extend to the contents of the books. 'Higher Algebra' to Böcher was necessarily something very different from 'the Modern Higher Algebra' to Salmon. In the interval between the two the sphere of Algebra had greatly widened, and the study of it had deepened. It is thus not surprising to find, so far as our subject is concerned, that the two books—if it be the

latest edition of Salmon that we use—are strikingly unlike. Whereas in Salmon six chapters out of twenty are specifically devoted to determinants, in Böcher there is only one out of twenty-two, although, it is true, much additional material is included in less compact form. On the other hand there are in Böcher two markedly useful chapters on 'Linear Equations', one on 'The Rank of a Determinant', and a still more important one on 'Invariant Factors'; whereas in Salmon there are to these practically no counterparts at all. A similar contrast, too, exists in regard to Cayleyan Matrices, which are not mentioned in Salmon, but which in Böcher receive considerable intermittent attention quite adequate for a well-filled chapter. In spite of all this, and of course greatly because of it, the new 'Introduction' is a most worthy companion to the old.

BOUKREIEFF, B. (1907, 1914): SHIFF, V. J. (1907)

[Elements of the Theory of Determinants (In Russian). *Bulletin . . . Univ. of Kieff.* Ann. 1907, pp. 1–50, 55–99. Also separately, iv + 100 pp. 2nd ed. vii + 128 pp. Kieff.]

[Elementary statement of some theorems in determinants (In Russian). 18 pp. St. Petersburg.]

In the first of these a little over fifty (50) pages are devoted to the properties of general determinants, and about twenty (20) to special determinants—axisymmetric, skew, Wronskians, Jacobians, Hessians. Two chapters (pp. 60–76) deal with the fundamental algebraical applications, and the last chapter (pp. 89–96) is historical.

PRANG, C. (1908): DÖLP, H. (1908)

[Determinanten. I. Hauptsätze über Determinanten. II. Einleitung in die analytische Geometrie . . . Zweite Auflage. vi + 65 pp. Berlin.]

[Die Determinanten. . . . 7 Aufl. ii + 95 pp. Darmstadt.]

The first of these is an improved recast of the former edition, with the contents separated as the title indicates.

The second is merely another reprint.

FISCHER, P. B. (1908)

[Determinanten. 134 pp. Leipzig.]

This belongs to the Göschen Collection of pocket booklets on science, and recalls a similar item in the Van Nostrand Series (*Hist.*, iv. p. 92). The page, however, is not quite so small, being more like that of the Hoepli Manuals (*Hist.*, iv. p. 94) and equally well filled.* The writer's plan and execution are commendable: in some essentials he is perhaps a trifle lavish,—for example, five different designations for one form of determinant seems a heavy load for so small a carrier.

KOWALEWSKI, G. (1909)

[Einführung in die Determinantentheorie. v + 550 pp. Leipzig.]

The quite unusual size of this textbook is in great part due to the considerable amount of space taken up with applications and to the exceptional treatment of one or two specialized determinants. The first six chapters (pp. 1–77) are occupied with the properties of general determinants, including the basic connection between determinants and linear equations. The next four (pp. 78–177) treat of five special forms, namely, compound †, bordered, axisymmetric, skew, orthogonant. Three chapters, largely applicational, are then, as it were, interjected (pp. 178–289), bearing the headings ' resultants and discriminants ', ' linear and quadratic forms ', and ' invariant factors '. These are succeeded by a return to the consideration of special forms—Jacobians, Wronskians, and multilineants, the last being a surprisingly long chapter (pp. 369–455). There then remain only a chapter on ' geometrical applications ' and two with the new titles ' linear integral-equations ' and ' Hilbert's special functions of a real symmetric kernel '. These two last have to be noted as the innovations among the nineteen; and there is the further striking

* Here again the low price (80 Pf.) calls for note.

† The assignment of a separate chapter to compound determinants is a new and helpful change in the arrangement of contents, the title ' zusammengesetzte ', however, not being taken over from Reiss but the paraphrase ' Determinanten deren Elemente Minoren einer andern sind ' substituted.

fact in regard to them that they occupy almost a sixth part of the whole work.

NETTO, E. (1910/5)

[Die Determinanten. vi + 129 pp. Leipzig.]

This is another textbook belonging to a series, the expected readers of the series being now engineers among others. Through increase in size of page it contains about a half more matter than Fischer's of 1908, with, of course, a corresponding increase in comprehensiveness, and it is at least as commendably planned and executed. Of its twelve chapters two of the less common may be singled out, namely, a chapter devoted to instruction in evaluation, and a chapter on arrays and the (non-zero) rank of an array. The properties are clearly enunciated throughout, and are well and fully illustrated. The explanations at times suggest that the writer had in view a book for self-instructors; and, whether this be so or not, such readers would find it useful.

LOEWY, A. (1910)

[Kombinatorik, Determinanten und Matrices. *Pascal's Repertorium d. höheren Math.* 2te Auflage, i. pp. 43–167.]

The amount of space devoted to determinants in this second German edition is almost treble that so assigned in the first Italian edition of 1898: the change, in fact, is comparable with the corresponding change in the *Encyklopädie d. math. Wiss.* from the German original of 1898 to the French edition of 1904. Of the latter work, however, we cannot say, as we may justly say of the *Repertorium*, that the second edition is in reality a different book. Pascal's original dwarf-leaved pocket volume disclaimed all pretensions to be encyclopaedic: its great usefulness and its acceptability lay in being a 'students' collection of results', somewhat resembling 'Carr's Synopsis' of 1886. The editor of the so-called new edition, while making the same profession, had found it difficult to guide his team of contributors in the intended path, with the result that the character and plan of the undertaking have suffered a considerable change along with the bulk. It is not unnatural, therefore, to find oneself com-

paring the contribution of Vogt in the *Encyclopédie* with that of Loewy in the *Repertorium*. For reference purposes the latter, like the former, is an improvement on its predecessor if only because of the increase in fullness; but equally with the former it stands in need of critical revision. Out of the two Vogt and Loewy ought to be able to produce a better third, and certainly a third showing greater accuracy in regard to historical matters.

Two fresh features that deserve special mention are a section (§ 6, pp. 79–92) on Cayleyan matrices, and another (§ 7, pp. 93–101), connected therewith, on Higher Complex Numbers. We may also note as being exceptionally full the section on Invariant Factors, and a section of miscellaneous content whose basic subject is Compounds, or, as they are called, 'abgeleitete Matrices'* in keeping with Cauchy's 'systèmes dérivés' of 1812.

In this matter of choice of contents it is interesting and not unprofitable to compare Kowalewski's nineteen chapters of 1909 with Loewy's seventeen sections of 1910, observing how differently the two authors map out the domain of their common subject, and how each one crosses over the boundary to strips of adjacent territory which the other passes by as having no attraction for him.

LOEWE, E. (1912): KAGAN, W. (1912)

[Die Verwendung der Determinanten im Unterricht der Höheren Schulen. Sch. Progr. 71 pp. Cöln.]

[Elements of the theory of determinants (In Russian). . . . pp. Odessa.]

The first of these is an experienced teacher's introduction to determinants. He devotes to the second order 10 pages, to the third 33 pages, to the fourth 24 pages, and save by implication nothing to the n^{th} order. In each case a start is made with the corresponding set of linear equations, the results arrived at being at once applied to analytical geometry. The copiousness of these applications is the outstanding feature of the booklet.

* The author, when speaking of H. J. S. Smith in this connection, not only omits to note that he (Smith) also uses the term 'derived matrix', but attributes to him instead the word 'concomitant' in an unheard-of sense.

NETTO, E. (1912): FISCHER, P. B. (1912): LIT, R. R. (1913): DÖLP, H. (1913)

[Die Determinanten. Russian translation, edited with notes by S. O. Shatunofsky. viii + 156 pp. Odessa.]

[Determinanten. 2te verbesserte Auflage. 136 pp. Berlin.]

[Beginselen van de Leer der Determinanten. 2de Uitgave. 80 pp. Amsterdam.]

[Die Determinanten, . . . 8 Aufl. iv + 95 pp. Darmstadt.]

In the first of these the increase in the number of pages is not owing to new matter, and the footnotes of the Russian editor are not at all extensive.

Although improvements have been made in the second, the work is substantially the same as before.

Save for a few additional exercises at the close the third is a reprint; and the fourth is a reprint entirely.

CALDARERA, F. ($1913^2/_1$)

[Trattato dei Determinanti. 255 pp. Palermo.]

This is a good-sized textbook in which the pure theory receives a very considerable share of attention, only about 50 pages being given up to applications, 20 to algebra, and the rest to geometry. Some special forms of determinants do not receive as much attention as in textbooks of similar scope; but this is counterbalanced by an exceptionally full chapter (pp. 169–202) on determinants of infinite order. A table is given (p. 74) of the values of the first fourteen of the peculiar centrosymmetric determinants dealt with in his paper of 1886 (*Hist.*, iii. p. 478). It is all the more regrettable to find the author taking over Dostor's conception of ' déterminants multiples '.

CULLIS, C. E. (1913)

[Matrices and Determinoids. Vol. I. xii + 430 pp. Cambridge.]

This handsome volume, the author tells us at the outset, " deals with *rectangular matrices* and *determinoids*, as distinguished from *square matrices* and *determinants*, the determinoid of a [non-quadrate] rectangular matrix being related to it in the same way as a determinant is related to a square matrix ". The idea of instituting such an analogue to a determinant must have occurred to many students of mathematics. Probably the least well-advised of them was Dostor, who made the innovation of calling

$$\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|$$

a ' déterminant multiple ', and of defining it as the equivalent of

$$\left| \begin{matrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right| .$$

Although the " determinoid " of Cullis

$$\left| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right|$$

has the same meaning, it is not at all because he proceeds in the same way. He defines a "determinoid" quite independently of a determinant, namely, as an aggregate of products of elements of an array, and he gives a rule for the formation of the products and a rule of signs: he then proceeds to elaborate the consequences of his definition. As a determinoid is a special case of a determinant, it is clear that this definition must include and degenerate into the usual definition of the former.

The book contains in all eleven chapters. The first (pp. 1–21) is generally introductory, and deals in particular with definitions and notations connected with the new entities: the second, third, and fourth extending in all to as many as eighty-two pages, deal with questions of signs: the fifth (pp. 105–152) concerns the various expansions of a determinoid: the sixth (pp. 153–208), a valuable chapter, treats of matrices in the Cayleyan sense—

the equality, addition, subtraction, and multiplication of them: the seventh (pp. 209–247) deals with the determinoid of the products considered in the sixth: the eighth concerns matrices whose elements are minor determinants of a fundamental matrix: the ninth and tenth—valuable chapters quite independently of determinoids—concern the rank of a matrix and the solution of matrix equations of the first degree: and the eleventh (pp. 304–417) contains a full and fresh treatment of the solution of a set of ordinary linear equations, the set being viewed as a matrix equation of the first degree.

In form the exposition is elaborately logical. It also bristles with words and phrases used in unfamiliar ways; all of these are carefully defined, and there is, in addition, a helpful Index, twenty-two columns in extent, which can be used as a Glossary.

Unlike the old textbooks on determinants this on determinoids has only one chapter that can be viewed as treating of an ‘ application ’, namely, the last. The author promises, however, to follow up his present work with two other volumes, the greater part of which would be occupied in remedying this seeming defect, the subjects to be dealt with being ordinary algebra, the algebra of quantics, algebraic geometry, and vector analysis. The appearance of these additional volumes will be looked forward to with much interest as they can scarcely fail to be of considerable service in the advancement of science. It will also then be apparent to what extent the theory of determinoids is likely to contribute towards such advancement. So far as one can at present see, there is safety in hazarding the conjecture that the extent will be limited as compared with that of its fellow-subject Matrices, or its prototype Determinants.

DICKSON, L. E. (1914): PETROVITCH, S. G. (1914) (?)

[Elementary Theory of Equations. v + 184 pp. New York.]

[On determinants (In Russian). 2nd ed. 49 pp. St. Petersburg.]

Chapter XI (pp. 127–149) of the first textbook here is devoted to determinants, and Chapter XII (pp. 150–166) to resultants and discriminants. Sets of appropriate exercises are inserted at intervals, thus enhancing the book for teaching purposes.

BĬUŜGENS, S. S. (1915): VINOGRADOV, S. P. (1915)

[Theory of Determinants (In Russian). 100 + 111 pp. Moscow.]

[Elements of the Theory of Determinants (In Russian). iv + 111 pp. Moscow.]

TOLEDO, L. O. DE (1916)

[Elementos de Aritmética Universal. Tome II. 421 pp. Madrid.]

More than a third of this volume (pp. 231–280) is devoted to determinants, and the section on the subject is preceded by another of almost like extent dealing with combinations, permutations, and allied matters. Of the seven chapters allotted to determinants, one treats of those of three dimensions. The connection of the general subject with linear equations is not referred to.

CULLIS, C. E. (1918)

[Matrices and Determinoids. Vol. II. xxiv + 555 pp Cambridge.]

This second handsome volume, more bulky than the first, hardly looks like a continuation although its eight chapters are numbered XII–XIX. The term ‘determinoid’ is almost entirely absent from it, whereas determinants, as even the Index makes clear, are under frequent discussion. This is more particularly manifest in the first four chapters, whose headings therefore it is well for us to note. They are: Chapter XII (pp. 1–36) on ‘Compound Matrices’, Chapter XIII (pp. 37–106) on ‘Relations between the elements and minor determinants of a matrix’, Chapter XIV (pp. 107–164) ‘Some properties of square matrices’, Chapter XV (pp. 165–227), ‘Ranks of matrix products and matrix factors’. If one word be wanted to describe roughly the contents of the volume as a whole, that word is *Matrices*. In comparison with the first volume the present seems less carefully planned and arranged: and, much to our regret, it shares with that

volume the serious fault of making no attempt to mark off new results from old, bibliographical and historical references being conspicuous by their absence.

BOSE, A. C. (1919)

[Cullis' Matrices and Determinoids. *Bull. Calcutta Math. Soc.*, x. pp. 243–256, xi. pp. 51–82.]

Ordinarily a review of a book would not find a place in our list of writings. An exception is made here because of the special character of the book reviewed, and because in such a case forty-six (46) pages of review from one who is both an expositor and a commentator could hardly fail to be useful.

CHAPTER III

Determinants and Linear Equations, from 1878 to 1918

The number of writings to be reported on here is almost exactly the same as for the immediately preceding period: in the quality of their contents, too, there is no appreciable difference.

FROBENIUS, G. (1878/$_4$)

[Theorie der linearen Formen mit ganzen Coefficienten. *Crelle's Journ.*, lxxxvi. pp. 146–208.]

In dealing with this paper in its proper place (*Hist.*, iv. pp. 444–445) we merely noted in a passing word that linear equations and congruences were treated at some length, more than this not being deemed necessary in view of the restriction made in it to coefficients and solutions that were *integral.* There is one of the theorems, however, that calls now for attention as being a noteworthy precursor of Capelli's of 1888 (*Hist.*, iv. pp. 104–105). It is the fourth theorem of § 8 (pp. 168–173) and in effect it stands as follows: *In order that a set of non-homogeneous linear equations may be satisfied by integral values of the unknowns, it is necessary and sufficient that the rank* r *of the unaugmented array of coefficients and the highest common divisor of the* r*-line minors of this array do not change when the augmented array is taken instead.* It will be observed how very little change is required to pass from Frobenius' wording to Capelli's, and how easy it would have been for Frobenius to justify it.

Having thus been fair to Frobenius we must allow him in his turn to be the same to H. J. Stephen Smith, who, he states, had in 1861, when as yet the only 'rank' thought of was the highest, formulated the criterion: "A linear system is, or is not, resoluble in integral numbers, according as the greatest-common-divisor of the determinants of the matrix of the system is, or is

not, equal to the corresponding greatest-common-factor of its augmented matrix." Oddly enough, before his communication had been published in full in the *Philosophical Transactions*, Smith had an experience similar to that of Frobenius, and as a consequence had used a footnote to page 310 of his memoir (*Coll. Math. Papers*, i. p. 387) to state that the criterion had already been given in the *Wiener Denkschriften* of 1856 by I. Heger (*Hist.*, ii. p. 92).

PRANGE, O. (1889)

[Lehrbuch der Gleichungen des ersten Grades mit mehreren Unbekannten. viii + 331 pp. Bremerhaven.]

This is a quite elementary textbook prepared on the so-called Kleyer system for self-instruction, and thus in form closely similar to Weichold's book on determinants already referred to (*Hist.*, iv. p. 92).

AURIC, A. (1892): MEYER, A. (1894)

[Les équations linéaires et leurs applications. Thèse. 81 pp. Paris.]

[Ligninger af 1ste Grad. *Nyt Tidsskrift f. Mat.*, B v. pp. 4–17.]

'ANON.' ($1897^1/_1$): GASCÓ, L. G. ($1897/_7$): FONTENÉ, G. ($1900^1/_4$)

[Note sur les équations linéaires. *Revue de Math. Spéc.*, iv. pp. 81–84.]

[Resolución por determinantes de los sistemas de equaciones lineales. *Archivo de Mat.*, ii. pp. 124–127.]

[Réclamation à propos du théorème dit "de Rouché". *Nouv. Annales de Math.*, (3) xix. p. 188.]

The first of these notes is in effect a proposed alternative proof of Rouché's theorem of 1875. The object of the third is to suggest that the said theorem should bear the name of Fontené as well,

it being assumed apparently that Fontené's original paper was not essentially different from Rouché's (*Hist.*, iii. pp. 90–92).

The second gives a simple alternative mode of arriving at the so-called Kramer's rule.

HEAWOOD, P. ($1900/_5$)

[On the fundamental proposition connected with the vanishing of a determinant. *Math. Gazette*, i. p. 344.]

The proposition in question is that the vanishing of the determinant of a set of linear homogeneous equations is the necessary and sufficient condition that there may be for the unknowns a set of satisfying values which are not all zero: and it is the second part of this which the writer very properly seeks to establish rigorously.

MUIR, T. ($1900^3/_{12}$)

(See under this heading in Chap. V.)

TWEEDIE, C. ($1900^{17}/_{12}$)

[Note on Dr. Muir's paper on a peculiar set of linear equations. *Proceed. R. Soc. Edinburgh*, xxiii. pp. 261–263.]

The expansion of the second of Muir's two determinants is here found in the same way as the first, the general form of each expansion being first arrived at, and then the values of the coefficients ascertained (*Hist.*, iii. p. 128).

GELIN, E. ($1901/_2$)

[Su di un sistema di equazione del primo grado. *Le Mat. pure ed appl.*, i. pp. 16–18, 25–29; or *Mathesis,* (3) i. suppl., 8 pp.]

The system in question is that which is sometimes named after Binet, whose treatment of it we have already fully explained (*Hist.*, ii. pp. 156–158). The present paper has an unusual wealth of deductions, and a more general set of equations is also fully dealt with.

GIUDICE, F. (1903)

[Sui sistemi lineari d'equazioni algebriche. *Giornale di Mat.*, xli. pp. 207–208.]

Here there is a discussion of the determinant

$$\begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1,\,m+n} \\ \cdot & \cdot & \cdot & \cdot \\ u_{m,\,1} & u_{m,\,2} & \cdots & u_{m,\,m+n} \\ v_{11} & v_{12} & \cdots & v_{1,\,m+n} \\ \cdot & \cdot & \cdot & \cdot \\ v_{n1} & v_{n2} & \cdots & v_{n,\,m+n} \end{vmatrix}$$

whose first m rows form the array (non-evanescent) of a set of linear homogeneous equations, and whose last n rows form a fundamental system of solutions of the said set of equations. Oddly enough, the discussion originates in connection with an extension of the self-evident proposition that *If any solution of the homogeneous set of equations*

$$a_{r1}y_1 + a_{r2}y_2 + \ldots + a_{r,\,m+n}\, y_{m+n} = 0 \Big\}_{r=1}^{r=m}$$

be $\eta_1, \eta_2, \ldots, \eta_{m+n}$, *and one particular solution of the corresponding non-homogeneous set*

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{r,\,m+n}\, x_{m+n} = u_r \Big\}_{r=1}^{r=m}$$

be $X_1, X_2, \ldots, X_{m+n}$, *then another solution of the latter set is*

$$\eta_1 + X_1,\ \eta_2 + X_2,\ \ldots,\ \eta_{m+n} + X_{m+n}.$$

A ' fundamental ' system of solutions, be it noted, is a mutually independent system from which by linear combination every solution is obtainable.

PEREZ, E. H. (1904)

[De algunos sistemas particulares de ecuaciones lineales. *Revista trim. de Mat.*, iv. pp. 24–28, 75–77.]

This is a sort of study of linear equations, ' whose coefficients are in arithmetical progression '. Although there is considerable variety in the determinants of the sets, there is nothing of special interest to note.

FROBENIUS, G. (1905/$_7$)

[Zur Theorie der linearen Gleichungen. *Crelle's Journ.*, cxxix. pp. 175–180.]

It would seem as if part of Frobenius' object here were to make clear his connection with two theorems in the Theory of Linear Equations. The first is that properly associated by him with Fontené and Rouché (*Hist.*, iii. pp. 90–92), the paper specified as containing his own contribution being that of 1876 (*Hist.*, iii. pp. 62–64, 275–277). The second, which he states at some length in an unformulated way, may without appreciable loss be condensed thus: *If the* r *linear homogeneous equations*

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = 0 \Big\}_{r=1}^{r=r}$$

be independent, and have the s *independent solutions*

$$(b_{11}, b_{12}, \ldots, b_{1n}),\ (b_{21}, b_{22}, \ldots, b_{2n}),\ \ldots \quad (b_{s1}, b_{s2}, \ldots, b_{sn}),$$

then the r-*line minors of the* a *array are proportional to the complementary* s-*line minors of the* b *array.* To this no name other than his own is attached, but it will be recognized as having been at least foreshadowed by Brill, Gordan, and d'Ovidio (*Hist.*, iii. pp. 35–37, 48–49, 92–93). Two arrays related as the a and b arrays are in the theorem just stated Frobenius speaks of as being 'adjugate', and he establishes the following property regarding them: *If the* n *columns of each array be separated into* r′ *columns followed by* s′ *columns, and the* r-*by*-r′ *array so obtained be of rank* ρ *and the* s-*by*-s′ *array be of rank* σ, then

$$\rho - \sigma = r - s' = r' - s.$$

From this, it is interesting to note, he deduces the test for the solvability of non-homogenous linear equations.

BES, K. (1905–1910)

[Uit de theorie der algebraische vergelijkingen. *Wiskundig Tijdschrift*, ii. pp. 2–16, . . . , vii. pp. 13–24. Also, separately, 166 pp. Haarlem.]

There is here (pp. 18–37) a freshly written chapter on our

present subject. Almost the whole of it is occupied with m linear homogeneous equations in n unknowns, the third case ($m < n$) being subdivided into three: (a) where all the minors of the array are different from 0, (b) where only some are different, (c) where all of them vanish.

MUIR, T. ($1906^{28}/_{2}$)

[A set of linear equations connected with homofocal surfaces. *Transac. S. African Philos. Soc.*, xvi. pp. 263–265.]

The set of equations referred to is that spoken of above under Gelin. Muir's object here is to draw attention to the striking applicability of a general method explained in an overlooked paper of Murphy's of 1832 (*Hist.*, ii. pp. 154–155). Other sets of equations similarly solvable are also given, and though the method is non-determinantal, the solutions obtained by it, of course, throw light on the determinants of the sets.

DUMAS, G. ($1906^{15}/_{11}$)

[Démonstration d'une proposition relative aux équations linéaires. *L'Enseignement Math.*, viii. pp. 448–455.]

The proposition in question is not distantly connected with Frobenius' first theorem of the preceding year. A little condensed in expression it may be stated thus: *If the* r *linear homogeneous equations*

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = 0 \Big\}_{r=1}^{r=r}$$

have the s *solutions*

$$(b_{11}, b_{12}, \ldots, b_{1n}), (b_{21}, b_{22}, \ldots, b_{2n}), \ldots, (b_{s1}, b_{s2}, \ldots, b_{s,n}),$$

then

$$\left.\begin{array}{l} x_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + \ldots + b_{s1}\gamma_s \\ x_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + \ldots + b_{s2}\gamma_s \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_n = b_{1n}\gamma_1 + b_{2n}\gamma_2 + \ldots + b_{sn}\gamma_s \end{array}\right\}$$

is a solution; and it includes every solution if neither the a *array*

nor the b *array be evanescent.* The theorem is newer in form than in substance, and the case $r = n - 1$ dates back to 1877 or earlier (*Hist.*, iii. pp. 92–93).

OCCHIPINTI, R. (1907)

[Sistemi lineari determinati od indeterminati le cui soluzioni constituiscono progressione aritmetica, geometrica od armonica. *Rivista di Fis., Mat. e Sci. Nat.*, viii. pp. 278–289.]

By ' solutions ' is here meant values of the unknowns that go to make a solution: and the author's object is to formulate a condition for a set of linear equations having a solution of one of three specified types. His starting-point is the determinantal values of the unknowns: and, as will be foreseen, the condition obtained is expressed in terms of minors of the augmented array of the set.

BÖCHER, M. (1907)

[Fundamental systems of solutions of homogeneous linear equations. *Introduction to Higher Algebra*, pp. 49–53.]

' Fundamental ' is used here in the special technical sense which we have already observed in Giudice's note of 1903, and which we have elsewhere incidentally tried to convey by the translation ' all-sufficient '. A textbook exposition of the term and of the facts that had gathered round the usage of it had become a desideratum and this Böcher most suitably supplies.

MÉRAY, C. (1907/$_{10}$)

[Sur la discussion et la résolution des équations simultanées du premier degré. *L'Enseignement Math.*, ix. pp. 337–366.]

Notwithstanding the difference in the title this new paper of Méray's has on the whole the same effect on the reader as his long exposition of 1884 (*Hist.*, iv. pp. 23–24). Properly enough his attitude towards the extreme determinantalist remains unchanged: but unfortunately he also continues to cherish some of his rival's prominent weaknesses. Of his thirty new pages twenty-four are preparatory to the business specified in the title,

'coevanescents' being the main subject; the remainder deals with sets of linear equations by what we may call the method of 'reduction'.

Oddly enough, the last paragraph is occupied with the unexpected theorem 'Tout déterminant est un polynome premier', his mode of proof resembling Hensel's of 1903.

KARST, L. (1909/$_4$)

[Lineare Funktionen und Gleichungen. Sch. Progr. 44 pp. Lichtenberg bei Berlin.]

This pamphlet belongs to the superior type of school-program, the author being not only painstaking but capable and well-informed. It is composed of nine short chapters or sections, beginning with one on dependence and independence of arrays and of linear functions (§ 1, pp. 3–9) and ending with one on the rank of the product of two arrays (§ 9, pp. 41–44). All the results arrived at in it, 53 in number, are singled out and carefully formulated: it thus may serve the purpose of a compendium as well as of a monograph. In quality it compares not unfavourably with Garbieri's well-known memoir of 1891 (*Hist.*, iv. pp. 105–106). This it also resembles in the attention which it gives to non-zero rank, the evanescence of arrays and other collateral matters. Its value however would have been greatly enhanced if the original sources used by the writer had been carefully specified. As it is, not a single such reference is given: and it is rare indeed that an author's name is mentioned helpfully at all. An exceptional instance is the calling of a theorem in passing the 'Grassmann-Brill' theorem, but even this, of course, only serves to whet the appetite.

KOWALEWSKI, G. (1909)

[Systeme linearer Gleichungen. *Einführungen in die Determinantentheorie*, pp. 45–65.]

This chapter (V) includes four theorems on the subject preceded by three theorems on the non-zero rank of an array. Of one of the four, Capelli's test for consistency (*Hist.*, iv. pp. 104–

105) a second proof is given. Frobenius' mode of arriving at a so-called 'fundamental' or all-sufficient set of solutions is reproduced, the initial step being to augment by $n - r$ rows the r-by-n array of the given set of independent simultaneous equations in such a way that the determinant $|a_{rs}|_n$ of the resulting square array does not vanish, thus giving for the desired set of solutions

$$\begin{array}{llll} A_{r+1,1}, & A_{r+1,2}, & \ldots, & A_{r+1,n} \\ A_{r+2,1}, & A_{r+2,2}, & \ldots, & A_{r+2,n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ A_{n,1}, & A_{n,2}, & \ldots, & A_{n,n}\ . \end{array}$$

where $|A_{rs}|_n$ is the adjugate of $|a_{rs}|_n$.

It has to be noted also that in the chapter on skew determinants two special sets of linear non-homogeneous equations are carefully discussed—the set (§ 63) whose determinant is skew and non zero-axial, and the set (§ 65) whose determinant is skew and odd-ordered. The discussion recalls one's mind to the origin of Pfaffians (*Hist.*, i. pp. 401–404).

MILLER, G. A. (1910/$_6$): RANUM, A. (1910/$_8$)

[On the solution of a system of linear equations. *American Math. Monthly*, xvii. pp. 137–139, 201–202.]

[On the classification of systems of linear equations. *American Math. Monthly*, xvii. pp. 155–161.]

After drawing attention deservedly to Capelli's test for consistence (*Hist.*, iv. pp. 104–105) the first author here proceeds to illustrate and establish the theorem that *a necessary and sufficient condition for a given unknown in a consistent set of linear equations having only one value is that the rank of the unaugmented array be lowered by omitting from it the coefficients of the said unknown.* For example, in the set

$$\left.\begin{aligned} 2x - \ \ y - 2z &= \ \ \ 8 \\ 4x - 2y - \ \ z &= -4 \\ 6x - 3y + \ \ z &= \ \ \ 4 \end{aligned}\right\}$$

where the rank of both unaltered arrays is 2 and the rank of the

narrower array got by withdrawing the coefficients of z is 1, the only value of z is 4.

This is generalized in the interesting paper of the second author as follows: *A necessary and sufficient condition that in all the solutions of a consistent set of linear equations in* h + k *unknowns the values of* $x_{h+1}, x_{h+2}, \ldots, x_{h+k}$ *shall be connected by a constant linear relation with coefficients not all zero is that the rank of the unaugmented array of the set be higher than the rank of the narrower array got by leaving out the coefficients of the said* k *unknowns.* It is then put to use for the purpose of obtaining a complete classification of sets of consistent linear equations: and finally the significance of the classification is pointed out when the x's are Cartesian co-ordinates of points in space of n dimensions.

RAMANUJAN, S. ($1912/_6$)

[Note on a set of simultaneous equations. *Journ. Indian Math. Soc.*, iv. pp. 94–96: v. pp. 61–62.]

An interesting non-determinantal solution of the classic set of equations met with by Sylvester in 1851 in dealing with the canonization of odd-degreed binary quantics (*Hist.*, ii. pp. 332–335). The editor reproduces the determinantal solution for comparison.

DOUGALL, J. ($1912/_{10}$, $1913/_1$)

[On the solubility of linear algebraic equations. *Math. Notes* (of Edinburgh Math. Soc.), i. pp. 125–129, 137–139.]

[Proof of the sufficiency of the determinant condition for the consistency of a system of n homogeneous linear equations in n variables. *Math. Notes*, i. pp. 139–140.]

The first note is a clear and incisive statement of the essentials of the subject, a contribution to verbal and logical accuracy

The gradational proof given in the second note is simple and effective, the specific object sought being the establishment of the existence of a non-null solution.

MOULTON, F. R. (1913$^{19}/_{5}$)

[On the solutions of linear equations having small determinants. *American Math. Monthly*, xx. pp. 242–249.]

Linear equations of the type here referred to have only once before been brought under our notice, namely, in Jürgens' pamphlet of 1886 (*Hist.*, iv. pp. 101–102). They are exemplified by the set

$$\left.\begin{array}{l} \cdot 34622x + \cdot 35381y + \cdot 36518z = \cdot 24561 \\ \cdot 89318x + \cdot 90274y + \cdot 91143z = \cdot 62433 \\ \cdot 22431x + \cdot 23642y + \cdot 24375z = \cdot 17145 \end{array}\right\}$$

in which the constants are supposed to be got by means of observations and to be subject to possible errors not exceeding five units in the sixth fractional place. The first important fact brought to notice by the author is that solutions so diverse as

$$\begin{array}{lll} x = -\ 1\cdot 027066, & y = +\ 2\cdot 091962, & z = -\ 0\cdot 380515 \\ x = -\ 1\cdot 022773, & y = +\ 2\cdot 084125, & z = -\ 0\cdot 376941 \\ x = -\ 1\cdot 031229, & y = +\ 2\cdot 099457, & z = -\ 0\cdot 383879 \end{array}$$

all satisfy the given equations up to the last place—that is to say, that the solution is not determinate to five places but only to two. The reason for this is gone into at some length, and the decision is finally come to that the uncertainty in the result depends on the determinant of the set, and may in fact be said to become greater as the determinant becomes less.

GRIEND, J. v. d. (1913)

[Vraagstuk 225. *Wiskundig Tijdschrift*, ix. Oplossingen, pp. 29–31.]

Concerns two sets of linear homogeneous equations whose arrays, l-by-n in the one case and m-by-n in the other, are such that $l + m > n$ and the product of any row of the one by any row of the other is 0. With it may be compared Giudice's note of 1903 where $l + m = n$.

VOSS, A. ($1915^1/_5$)

[Ueber die Transformation linearen Formen und die Lösung linearen Gleichungen. *Sitzungsb.... Akad. d. Wiss.* (München), 1915, pp. 231–256.]

For the student of linear equations the interest here lies in the fact that the solution provided is *symmetrical* in the sense used in connection with Frobenius' solution of 1876,* and for the student of determinants merely in the skilful use made of them in expounding the solution in question.

KAPTEYN, W. (1915, 1918)

[Vraagstukken 19, 177. *Wiskundige Opgaven,* xii. pp. 46–47, 407–408.]

The two sets of equations here solved belong to the same type, being special instances of the general set whose determinant is $|a_1^1\, a_2^2 \, \ldots \, a_n^n|$ and which we have above spoken of as Newton's (*Hist.*, ii. pp. 154–155). In the one case the a's are the first n integers; in the other they are the n^{th} roots of 1; and in the latter there is the further specialization that all the right-hand members except one are 0.

GRADARA, V. (1916)

[Quelques considérations sur les systèmes de formes linéaires. *Annaes sci. da Acad. Polyt.* (Porto), xi. pp. 58–64.]

The considerations referred to concern the subject of 'independence' as defined and discussed in connection with the concept 'characteristic' or 'rank' from the time of Capelli and Garbieri onward (*Hist.*, iv. pp. 102–103). The writer aims merely at improvement in textbook exposition, singling out as an example the treatment of the theorem: *If in a set of* m *linear forms there be a set of* h *forms in terms of which the others can be linearly expressed, the characteristics of the two sets are identical.*

* *Crelle's Journ.*, lxxxii. pp. 230–315.

STUDY, E. (1918)

[Zur Theorie der linearen Gleichungen. *Acta Math.*, xlii. pp. 1–61.]

Very little of the interest attaching to this long memoir is determinantal. The author opens with the statement that 'determinants do not always give the most suitable solution'; and we soon learn from him not only a case in point but also of his mode of providing in that case a better solution. It must suffice therefore to say that in the very special set of equations with which he deals the coefficients and the unknowns are quaternions, and that not unnaturally the function which proves his best auxiliary is the so-called 'Nabla' (∇). In an appendix another special set of equations is considered: here, however, it is one of the oldest, Jacobi's of 1827 (*Hist.*, i. pp. 401–405, 263), and the new solution rests on the theory of quantics.

BURGESS, H. T. ($1918^{1}/_{12}$)

[Practical solution of linear equations. *American Math. Monthly*, xxv. pp. 441–444.]

The writer's object here is to supplement Böcher's exposition of 1907 by giving, he says, a readily applicable rule of procedure for finding a 'fundamental' system of solutions. Two simple examples are added in illustration, the system of solutions which is obtained in the one case consisting of two and in the other of one.

CHAPTER IV

AXISYMMETRIC DETERMINANTS, FROM 1900 TO 1918

For the period under consideration the number of writings on axisymmetric determinants is about a half more than for the period 1880–1897. It has to be noted, too, for the sake of readier reference, that the duplicant which in form is strictly axisymmetric, is not so classed, being assigned to Chap. I when its basic determinant is general, and elsewhere when not; for example, under Muir of $1914^{20}/_2$ in Chap. I and under Muir of $1914^{28}/_9$ in Chap. XI. For the same reason the seeker would do well to bear in mind that the axisymmetric, besides being like its fellow specials a particular case of $|a_{1n}|$, is included also as a case of the hermitant: for example, under Nicoletti of $1909^2/_5$ in Chap. XI.

NIELSEN, H. S. (1893)

[Opgave 178. *Nyt Tidsskrift f. Math.*, A iv. pp. 116–117.]

The novelty brought forward here is the fact that the equation

$$\begin{vmatrix} x & A & B \\ A & x & C \\ B & C & x \end{vmatrix} = 0$$

has 2 for a root when

$$A, B, C = m + m^{-1},\ n + n^{-1},\ mn^{-1} + m^{-1}n,$$

since then $ABC + 4 = A^2 + B^2 + C^2$.

KANTOR, S. ($1900^{13}/_1$)

[Ein Theorem über Determinanten. *Nachrichten . . . Ges. d. Wiss.* (Göttingen) 1899, pp. 272–281.]

The origin of this paper we have already referred to in our

first chapter. Most of the interest attaching to it is in connection with two papers of Frobenius (*Hist.*, iii. pp. 275–277; iv. p. 217), the determinants used in the one being mainly axisymmetric and in the other zero-axial skew. One of two points to be specially noted is that Kantor's proofs are based on geometrical considerations, and another is that in his results the two types of determinants are not separated, what is proved true of the one being true of the other. His first formulated result of this kind is that *If in an axisymmetric or zero-axial skew determinant all the* m-*line and* (m + 1)-*line coaxial minors vanish that contain a fixed* (m — 1)-*line coaxial minor, then all the* m-*line minors vanish* —a statement which so far as axisymmetric determinants are concerned had already been made by Frobenius (*Hist.*, iv. p. 135). His next is the related result: *If in an axisymmetric or zero-axial skew determinant the* m-*line minors do not all vanish, then must every non-zero* (m — 1)-*line coaxial minor be contained in a non-zero* m-*line or* (m + 1)-*line coaxial minor.* The two others are much more general, and are unavoidably more complicated in expression.

MUIR, T. (1900/3)

[On certain aggregates of determinant minors. *Proceed. R. Soc. Edinburgh*, xxiii. pp. 142–154.]

From a general result there is here (§ 5) readily deduced an amendment of Kronecker's theorem of 1882 (*Hist.*, iv. p. 113), namely, *If any even-ordered determinant* $\begin{vmatrix} 1 & 2 & 3 & \ldots & 2m \\ 1 & 2 & 3 & \ldots & 2m \end{vmatrix}$ *have its last* (m + 1)-*line minor axisymmetric, then*

$$\begin{vmatrix} 1 & 2 & 3 & \ldots & m-1 & m \\ m+1 & m+2 & m+3 & \ldots & 2m-1 & 2m \end{vmatrix}$$

$$-\begin{vmatrix} 1 & 2 & 3 & \ldots & m-1 & m+1 \\ m & m+2 & m+3 & \ldots & 2m-1 & 2m \end{vmatrix}$$

$$+\begin{vmatrix} 1 & 2 & 3 & \ldots & m-1 & m+2 \\ m & m+1 & m+3 & \ldots & 2m-1 & 2m \end{vmatrix} - \ldots = 0.$$

For example, when $\begin{vmatrix} 45678 \\ 45678 \end{vmatrix}$ is axisymmetric there exists the identity

$$\begin{vmatrix} 1234 \\ 5678 \end{vmatrix} - \begin{vmatrix} 1235 \\ 4678 \end{vmatrix} + \begin{vmatrix} 1236 \\ 4578 \end{vmatrix} - \begin{vmatrix} 1237 \\ 4568 \end{vmatrix} + \begin{vmatrix} 1238 \\ 4567 \end{vmatrix} = 0.$$

or, as it may conveniently be written (*Hist.*, iv. p. 24),

$$\begin{vmatrix} 123\bar{4} \\ \underline{5678} \end{vmatrix} = 0.$$

DEMOULIN, A. (1900/7)

[Question 1280. *Mathesis*, (2) x. p. 176; (3) i. pp. 230–233.]

In several ways the $(n+1)$-line determinant of the form

$$\begin{vmatrix} x_1^2+a & x_1x_2 & x_1x_3 & y_1 \\ x_1x_2 & x_2^2+a & x_2x_3 & y_2 \\ x_1x_3 & x_2x_3 & x_3^2+a & y_3 \\ y_1 & y_2 & y_3 & -1 \end{vmatrix}$$

is here evaluated. When n is 3 the result is

$$-a^3 - a^2\Sigma(x_1^2+y_1^2) - a\Sigma|x_1y_2|^2.$$

It is worth noting in connection therewith that

$$-\begin{vmatrix} -1 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & 1+a_1 & 1 & 1 \\ \xi_2 & 1 & 1+a_2 & 1 \\ \xi_3 & 1 & 1 & 1+a_3 \end{vmatrix} = \begin{vmatrix} -1 & . & 1 & 1 & 1 \\ . & -1 & \xi_1 & \xi_2 & \xi_3 \\ 1 & \xi_1 & a_1 & . & . \\ 1 & \xi_2 & . & a_2 & . \\ 1 & \xi_3 & . & . & a_3 \end{vmatrix}$$

$$= a_1a_2a_3 + \Sigma a_1a_2(\xi_3^2+1) + \Sigma a_1(\xi_2-\xi_3)^2.$$

HOFFBAUER, . (1900/7)

[Question 1885. *L'Intermédiaire des Math.*, vii. p. 236; xxii. p. 81.]

The moving idea here is essentially the same as Sylvester's of 1853 (*Hist.*, ii. pp. 127–129), but it is not faultlessly employed.

METZLER, W. H. (1901/$_3$)

[On certain aggregates of determinant minors. *Transac. American Math. Soc.*, ii. pp. 395–403.]

This is a carefully worked-out paper following up Muir's of 1888 and 1900, and makes a distinct advance under both special heads of the latter, axisymmetric and centrosymmetric. As regards the even-ordered axisymmetric (pp. 397–400) the new note to be taken is that Kronecker's is not the only type of vanishing aggregate associated with that determinant—that, in fact, Kronecker's is merely one of a family. Thus, recurring to the case where n is 4, we now learn of the family

$$\Sigma\begin{vmatrix}12\overline{3}4\\ \underline{5678}\end{vmatrix}=0,\quad \Sigma\begin{vmatrix}12\overline{34}\\ \underline{5678}\end{vmatrix}=0,\quad \Sigma\begin{vmatrix}1\overline{234}\\ \underline{5678}\end{vmatrix}=0,\quad \Sigma\begin{vmatrix}\overline{1234}\\ \underline{5678}\end{vmatrix}=0,$$

of which Kronecker's is the first. The theorem is enunciated in all its generality with the help of the author's own notation; and the basis of his proof is, as with Muir, a double use of Laplace's expansion-theorem.

NANSON, E. J. (1902/$_1$)

[A note on determinants. *Messenger of Math.*, xxxi. pp. 140–143.]

The object of this paper is the same as that of White's of 1895, namely, to show that the linear relation between m-line minors of a $2m$-line axisymmetric determinant can be deduced from Sylvester's theorem of 1851 regarding products of pairs of determinants. The method followed, however, is quite different from White's, though not more effective.

DIXON, A. C. (1902^{25}/$_1$)

[Note on the reduction of a ternary quantic to a symmetrical determinant. *Proceed. Cambridge Philos. Soc.*, xi. pp. 350–351.]

From geometrical considerations it is here shown that, for every theta function of even characteristic which does not vanish for zero values of the arguments, there is one reduction of the

ternary quantic to the form of an axisymmetric determinant with linear elements.

MUIR, T. ($1902^{20}/_1$)

[The applicability of the law of extensible minors to determinants of special form. *Proceed. Edinburgh Math. Soc.*, xx. pp. 44–49.]

Following up a suggestion of 1897 (*Hist.*, iv. p. 137) the author now fully establishes the following theorem regarding Kronecker's vanishing aggregate of minors of an axisymmetric determinant. *If each* m-*line minor in such an aggregate be written in the form*

$$\begin{vmatrix} a, & b, & c, & \ldots \\ \alpha, & \beta, & \gamma, & \ldots \end{vmatrix}$$

where a, b, c, . . . *are the numbers of the rows and* α, β, γ, . . . *the numbers of the columns which the minor occupies in the original axisymmetric determinant of the* $2\mathrm{m}^{th}$ *order, then a generalization (the Extensional) is obtained by changing*

$$\begin{vmatrix} a, & b, & c, & \ldots \\ \alpha, & \beta, & \gamma, & \ldots \end{vmatrix} \quad \textit{into} \quad \begin{matrix} a, & b, & c, & \ldots \\ \alpha, & \beta, & \gamma, & \ldots \end{matrix}$$

where the numbers now denote omitted lines in an axisymmetric determinant of any order higher than the $(2\mathrm{m}-1)^{th}$. It suffices to point out that the necessary requirements for effecting a proof are (*a*) that the extended determinant be of the same special form as the original and contain the latter as a minor, and (*b*) that the adjugate be a determinant of the same special form as the original.

MUIR, T. ($1902/_2$)

[Aggregates of minors of an axisymmetric determinant. *Philos. Magazine*, (6) iii. pp. 410–416.]

This paper follows intimately on Metzler's of the year before. Its main feature is the drawing of an important distinction between the new vanishing aggregates and the original, the nature of the distinction being something resembling that between Sylvester's

degrees of nullity (*Hist.*, iv. p. 17). For example, while it is true that

$$\Sigma \begin{vmatrix} 12\overline{34} \\ \underline{5678} \end{vmatrix},$$

one of the new aggregates, is null, it is further true that it is not a ‘ simple null ’, being the sum of four null sub-aggregates, namely,

$$\Sigma \begin{vmatrix} 123\overline{4} \\ \underline{5678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 12\overline{45} \\ \underline{3678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 13\overline{45} \\ 2\underline{678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 1\overline{456} \\ 23\underline{78} \end{vmatrix}.$$

The real interest thus lies in vanishing-aggregates that are *fundamental.* The case of the 8-line axisymmetric and the case of the 10-line are worked out in detail, it being shown among other things that the fundamental vanishing aggregates are

$$\Sigma \begin{vmatrix} 123\overline{4} \\ \underline{5678} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 12\overline{4}5 \\ 3\underline{678} \end{vmatrix},$$

and

$$\Sigma \begin{vmatrix} 123\overline{45} \\ \underline{67890} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 123\overline{56} \\ 4\underline{7890} \end{vmatrix}, \quad \Sigma \begin{vmatrix} 12\overline{567} \\ 34\underline{890} \end{vmatrix}$$

respectively.

RUSSIAN, C. K. ($1902^1/_3$)

[On a new property of symmetric determinants (In Russian). *Mem. . . . Soc. of Naturalists of New Russia* (Odessa), xxvi. pp. x-xv.]

The theorem here established concerning axisymmetric and zero-axial skew determinants is a rediscovery as regards the former (*Hist.*, iii. pp. 103–104).

MUIR, T. ($1902/_4$)

[Vanishing aggregates of secondary minors of a persymmetric determinant. *Transac. R. Soc. Edinburgh*, xl. pp. 511–533.]

In spite of the title the greater part (§§ 1–4, 15–32) of this lengthy paper is more closely concerned with axisymmetric than

with persymmetric determinants. The first theorem of any note is (§ 17): *If to* n — 2 *given columns having* n *elements each there be appended the* r^{th} *column of a given determinant of the* n^{th} *order, and from the resulting array the* r^{th} *row be deleted, there being thus produced a square array of the* $(n-1)^{th}$ *order the determinant of which is* D_r, *then the aggregate*

$$\sum_{r=1}^{r=n} (-1)^{r-1} D_r$$

will vanish when the given determinant is axisymmetric. For example, when the two given arrays are

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{array} \qquad \begin{array}{cccc} a_1 & \beta_1 & \gamma_1 & \delta_1 \\ a_2 & \beta_2 & \gamma_2 & \delta_2 \\ a_3 & \beta_3 & \gamma_3 & \delta_3 \\ a_4 & \beta_4 & \gamma_4 & \delta_4 \end{array}$$

the aggregate in question, say a 'Kronecker aggregate', is

$$| x_2 y_3 a_4 | - | x_1 y_3 \beta_4 | + | x_1 y_2 \gamma_4 | - | x_1 y_2 \delta_3 |$$

and may be conveniently denoted by

$$[x\ y;\ \ a_1 \beta_2 \gamma_3 \delta_4]$$

Following on this comes a series of such theorems of increasing generality but unfortunately not of decreasing complexity, all of them relating to an aggregate which vanishes when some connecting determinant becomes axisymmetric. In each case the vanishing is established by showing that the aggregate is expressible in terms of Kronecker aggregates of the type

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{vmatrix},$$

which in analogy with the above notation is temporarily denoted by

$$[1, 2;\ \ 3, 4, 5, 6].$$

For example, the aggregate

$$| x_1 y_2 a_4 \beta_5 \gamma_6 | - | x_1 y_2 a_3 \beta_5 \delta_6 | + | x_1 y_2 a_3 \beta_4 \epsilon_6 | - | x_1 y_2 a_3 \beta_4 \zeta_5 |$$

is shown to be equal to

$$\sum_{p<q}^{q=2,\ldots,6} (-1)^{p+q-1} \left| x_p y_q \right| \cdot \vdots\, p,\ q;\quad 3,\ 4,\ 5,\ 6\, \vdots,$$

which, by reason of the form of the second factors of its terms, is known to vanish when $|\alpha_1\beta_2\gamma_3\delta_4\epsilon_5\zeta_6|$ is axisymmetric.

MUIR, T. (1902[21]/5)

[Note on Kronecker's linear relation in determinants. *Messenger of Math.*, xxxii. pp. 4–6.]

Once again the subject is the possibility of using Sylvester's product-theorem of 1851 as a basis from which to deduce the linear relation connecting certain minors of an axisymmetric determinant. This time, however, the paper is critical rather than constructive, the object being to suggest the need for caution in seeking to establish new propositions by means of a method whose own logical soundness is not too patent.

It seems more important now to point out that none of the three writers concerned—White, Nanson, Muir—draw attention to the fact that the reverse order of logical succession is more easy to follow. Thus $r \cdot s$ standing for $a_r a_s + b_r b_s + c_r c_s$ the determinant

$$|1\cdot 1 \quad 2\cdot 2 \quad 3\cdot 3 \quad 4\cdot 4 \quad 5\cdot 5 \quad 6\cdot 6|$$

is axisymmetric: consequently from Kronecker

$$\begin{vmatrix} 1\cdot4 & 1\cdot5 & 1\cdot6 \\ 2\cdot4 & 2\cdot5 & 2\cdot6 \\ 3\cdot4 & 3\cdot5 & 3\cdot6 \end{vmatrix} - \begin{vmatrix} 1\cdot3 & 1\cdot5 & 1\cdot6 \\ 2\cdot3 & 2\cdot5 & 2\cdot6 \\ 4\cdot3 & 4\cdot5 & 4\cdot6 \end{vmatrix} + \begin{vmatrix} 1\cdot3 & 1\cdot4 & 1\cdot6 \\ 2\cdot3 & 2\cdot4 & 2\cdot6 \\ 5\cdot3 & 5\cdot4 & 5\cdot6 \end{vmatrix} - \begin{vmatrix} 1\cdot3 & 1\cdot4 & 1\cdot5 \\ 2\cdot3 & 2\cdot4 & 2\cdot5 \\ 6\cdot3 & 6\cdot4 & 6\cdot5 \end{vmatrix} = 0,$$

and from this comes by the multiplication-theorem

$$|a_1b_2c_3|\,|a_4b_5c_6| - |a_1b_2c_4|\,|a_3b_5c_6| + |a_1b_2c_5|\,|a_3b_4c_6| \\ - |a_1b_2c_6|\,|a_3b_4c_5| = 0,$$

that is to say, Bezout's instance of the so-called Sylvester theorem regarding products of pairs of general determinants.

ORLANDO, L. (1902)

[Relazioni fra i minori d' ordine p d' una matrice quadrata di caratteristica p. *Giornale di Mat.*, xl. pp. 275–277.]

As a deduction from a more general theorem there is here obtained Frobenius' first theorem of 1876 in regard to axisymmetric determinants (*Hist.*, iii. p. 275); also the further deduction that *In an axisymmetric determinant of non-zero rank* k *the non-vanishing* k*-line coaxial minors have all the same sign.*

MUIR, T. ($1902^{16}/_{7}$): NANSON, E. J. ($1902^{1}/_{12}$)

[The Jacobian of the primary minors of an axisymmetric determinant with reference to the corresponding elements of the latter. *Philos. Magazine*, (6) iv. pp. 507–512.]

[Question 15244. *Educ. Times*, lv. p. 517.]

The first of these will be found dealt with in the chapter on Jacobians.

The second seeks proof of the assertion that if all the coaxial minors of an axisymmetric determinant be positive, and P_r stand for the product of those of the r^{th} order, then

$$P_1 > (P_2)^{\frac{1}{(n-1)_1}} > (P_3)^{\frac{1}{(n-1)_2}} > \dots > P_n.$$

RUSSIAN, C. K. ($1903^{5}/_{1}$)

[Kilka twierdzén z teoryi wyznaczników. (Einige Determinantensätze). *Bull. . . . Acad. . . . Cracovie*, 1903, pp. 1–7: or *Rozprawy Akad. Umiejętności*, xliii. pp. 8–13.]

We are here presented with an interesting cluster of six theorems concerning any determinant that is either axisymmetric or zero-axial skew. The author follows up that portion of Frobenius' memoir of 1876 which deals with the same subject (*Hist.*, iii. pp. 62–64, 275–277). The first and fundamental theorem is: *If the sum of the diagonal elements of the adjugate of an axisymmetric or zero-axial skew determinant* Δ *be* 0, *then the sum of the squares of the elements of any row of the said adjugate contains* Δ

as a factor. The second is an easy deduction from it, namely: *If an axisymmetric or zero-axial skew determinant vanish and also the sum of its primary coaxial minors, then every one of its primary minors must vanish.* The third is an extension of the second, and the fourth is an extension of Frobenius' first (*Hist.*, iii. p. 275). Theorems of Frobenius are also generalized in the fifth and sixth, but here the generalization is wider, as in these cases Frobenius speaks of zero-axial skew determinants only (p. 276).

MUIR, T. (1903 13/7)

[The theory of axisymmetric determinants . . . up to 1841. *Proceed. R. Soc. Edinburgh*, xxiv. pp. 555–571.]

This is the first of our papers on the history of axisymmetric determinants, sketching the early foreshadowings by Lagrange and Gauss, and then dealing methodically with ten writings of which the main authors were Cauchy and Jacobi.

KÜRSCHÁK, J. (1903 20/7)

[Ueber symmetrische Matrices. *Math. Annalen*, lviii. pp. 380–384.]

The introduction to this paper is the statement that every linear function, F, of minors of the axisymmetric determinant $| y_{11}\, y_{22} \ldots y_{nn} |$ satisfies a certain set of homogeneous linear differential equations whose every differential-coefficient is of the form $\partial^2 F/\partial y_{ik}\partial y_{\mu\nu}$: and the rest of the paper is occupied in establishing two converse theorems.

CWOJDZIŃSKI, K. (1901, 1903): JUHEL-RÉNOY, J. (1908)

[Distanzrelation zwischen Punkten und Graden. . . . *Archiv d. Math. u. Phys.*, (3) v. pp. 118–122: ix. pp. 8–10.]

[Sur l'application des déterminants à la géométrie. *Nouv. Annales de Math.*, (4) viii. pp. 258–263, 450–456.]

The fresh contributions in these papers are mainly, if not wholly, geometrical.

NANSON, E. J. (1903/8)

[Minors of axisymmetric determinants. *American Journ. of Math.*, xxvii. pp. 69–76.]

What the author here gives is an elaboration of his note of the previous year, the object being to make clear the all-powerfulness of the method of derivation employed, namely, the passage from an equality regarding an aggregate of products of pairs of general determinants to an equality regarding an aggregate of minors of an axisymmetric determinant. One interesting point, not specially insisted on but none the less important, is the entire removal of the restriction usually made as to the order of the basic determinant (*Hist.*, iv. p. 137).

METZLER, W. H. (1905^13/3)

[Variant forms of vanishing aggregates of minors of axisymmetric determinants. *Proceed. R. Soc. Edinburgh*, xxv. pp. 717–721.]

The title here is fully descriptive of the contents. Metzler's recurrence to the subject was doubtless due to Nanson's paper of 1903 with its comments on vanishing aggregates other than Kronecker's: indeed he confesses that his main object is to show that Nanson had there reached nothing that could not be readily obtained from the vanishing aggregates which he (Metzler) had previously brought to notice. Whether this be so or not, we have at least the benefit of a freshly constructed restatement of views, just as Nanson's paper under reply had itself been.

METZLER, W. H. (1905^15/5)

[Vanishing aggregates of determinant minors. *Proceed. R. Soc. Edinburgh*, xxv. pp. 853–861.]

In the latter part of this attention is recalled to Muir's investigation of 1902 into the subject of Kronecker and other aggregates of determinants, the main matter considered being an aggregate which vanishes when axisymmetry is introduced into one of the two arrays from which the determinants of the

aggregate are formed. The author goes back to his very comprehensive theorem of 1901 (see Chap. I, above) and succeeds in deducing from it by specialization one of the most general of Muir's results.

GIAMBELLI, G. Z. ($1905^{19}/_{12}$): PETR, K. ($1906^{19}/_{1}$)

[Sulle varietà rappresentate coll' annullare determinanti minori contenuti in un determinante simmetrico od emisimmetrico generico di forme. *Atti . . . Accad. delle Sci.* (Torino), xli. pp. 102–125.]

[Die symmetrische Zahlensysteme und der Satz von Sturm. *Bull. internat. de l'Acad. des Sci.* (Prague), xi. pp. 14–34; or *Rozpr. České Akad.* . . ., xv. pp. 1–19.]

In neither of these papers is the author concerned with the advancement of the theory of axisymmetric determinants: indeed, viewed from our present standpoint the papers are purely applicational, and might have been relegated as in similar cases to an auxiliary list of titles at the end of the chapter. It deserves to be noted, however, that both are exceptional in the extent to which they make use of the determinants in question, and that consequently the chance of them proving suggestive is so much the greater. The subject of the first is as the title indicates,* and that of the second is the determination of the ' signature ' of an n-ary quadric. This latter, we may mention, attracted Frobenius and led him to write a related paper, known as his second paper on the Law of Inertia, in the Berlin *Sitzungsberichte* (1906, pp. 657–663).

MUIR, T. ($1906^{26}/_{2}$)

[A Pfaffian identity, and related vanishing aggregates of determinant minors. *Transac. R. Soc. Edinburgh*, xlv. pp. 311–321.]

The subject of aggregates of minors, so often referred to above, receives incidentally here (§§ 3, 4, 6) a curious contribution, namely, the expression of such an aggregate by means of Pfaffians. For example, Kronecker's aggregate

$$|a_4b_5c_6| - |a_3b_5d_6| + |a_2c_5d_6| - |b_1c_5d_6|$$

* See Auxiliary List, Chap. I.

connected with $|a_1b_2c_3d_4e_5f_6|$ and representable by

$$\Sigma\left|\begin{array}{ccc} 1 & 2 & 3 \\ 4 & \underset{\cdot}{5} & \underset{\cdot}{6} \end{array}\right|$$

is shown to be equal to

$$-\left|\begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & \cdot \end{array}\right| + \left|\begin{array}{ccccc} b_1 & c_1 & d_1 & a_5 & a_6 \\ & c_2 & d_2 & b_5 & b_6 \\ & & d_3 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & \cdot \end{array}\right|$$

and therefore to vanish when $|a_1b_2c_3d_4|$ is axisymmetric. The most general result is that which concerns

$$\Sigma\left|\begin{array}{cccc} 1 & 2 & \cdots & m \\ m+1 & m+2 & \cdots & 2m \end{array}\right|,$$

this being expressed by means of a Pfaffian whose elements are the differences of the conjugate elements of a determinant: for example,

$$-\Sigma\left|\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right| = \left|\begin{array}{ccc} a_2-b_1 & a_3-c_1 & a_4-d_1 \\ & b_3-c_2 & b_4-d_2 \\ & & c_4-d_3 \end{array}\right|,$$

$$-\Sigma\left|\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right| = \left|\begin{array}{cccc} a_2-b_1 & a_3-c_1 & \cdots & a_6-f_1 \\ & b_3-c_2 & \cdots & b_6-f_2 \\ & \cdot & \cdots & \cdot \\ & & & e_6-f_5 \end{array}\right|,$$

where again the effects of axisymmetry are at once apparent. As examples of specialized aggregates, other than Kronecker's, may be given

$$-\Sigma\left|\begin{array}{ccc} \dot{1} & 2 & 3 \\ 4 & 5 & 6 \end{array}\right| = \left|\begin{array}{ccccc} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3-c_2 & b_4-d_2 & b_5-e_2 & b_6-f_2 \\ & & c_4-d_3 & c_5-e_3 & c_6-f_3 \\ & & & d_5-e_4 & d_6-f_4 \\ & & & & e_6-f_5 \end{array}\right|$$

which is seen to vanish when any one of the 4-line coaxial minors of $| b_2 c_3 d_4 e_5 f_6 |$ is axisymmetric: and

$$-\Sigma \begin{vmatrix} \cdot & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 - c_2 & b_4 & b_5 - e_2 & b_6 - f_2 \\ & & c_4 & c_5 - e_3 & c_6 - f_3 \\ & & & - e_4 & - f_4 \\ & & & & e_6 - f_5 \end{vmatrix} .$$

which vanishes when $| b_2 c_3 e_5 f_6 |$ is axisymmetric.

FONTENÉ, G. (1906¹/₄)

[Question 2041. *Nouv. Annales de Math.*, (4) vi. p. 192; vii. pp. 47–48.]

The determinant here is Tirelli's second of 1875 (*Hist.*, iii. p. 459). We may now add that there is profit in looking on it as the cofactor of the $(1, 1)^{\text{th}}$ element of

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & r+1 \\ 1 & 3 & 6 & \dots & (r+2)_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & r+1 & (r+2)_2 & \dots & (2r)_r \end{vmatrix}, \text{ or } D \text{ say;}$$

for, this being known (*Hist.*, iii. p. 450) to be equal to 1, we know also that D with its $(1, 1)^{\text{th}}$ element changed into 0 is equal to $-r$.

COBLE, A. B. (1906¹/₁₁)

[The linear relations among the minors of symmetric determinants. *Johns Hopkins Univ. Circ.*, No. 191, pp. 86–90.]

The writer here takes a fresh view-point, calling to his aid the invariant theory of bilinear forms—the subject, be it noted, which Kronecker was engaged on when he originally lit upon the relations in question. The clearly established result is that *The linear relations among the minors of an* n-*line axisymmetric determinant are isomorphic with the quadratic relations among the co-*

ordinates of the various spaces in a space S_{n-1}. The paper and those of White (1895) and Nanson (1902) gain by juxtaposition.

TRACHTENBERG, H. L. ($1906^1/_{12}$)

[Question 16116. *Educ. Times*, lix. p. 540: lx. p. 81: or *Math. from Educ. Times*, (2) xii. pp. 43–44.] *

The property here proved is that

$$\text{if} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad \text{be such that} \quad \frac{F}{f} + \frac{G}{g} + \frac{H}{h} = \lambda,$$

$$\text{then } f\left(\frac{A}{f} + \frac{H}{g} + \frac{G}{h}\right) = \lambda^2.$$

MUIR, T. ($1906^{24}/_{12}$)

[The theory of axisymmetric determinants . . . up to 1860. *Proceed. R. Soc. Edinburgh*, xxvii. pp. 135–166.]

This, the second of our papers on the history of axisymmetric determinants, contains an account of twenty-six writings, many of them of considerable interest, by Cayley, Sylvester, Brioschi, Hesse, &c., and the account opening with Cayley's first paper of all, dated May 1841.

MUIR, T. ($1906^{22}/_{12}$, $1907^{25}/_1$): 'Un Abonné' ($1909^1/_4$)

[A property of axisymmetric determinants connected with the simultaneous vanishing of the surface and volume of a tetrahedron. *Transac. S. African Philos. Soc.*, xvi. pp. 445–457.]

[The norm which is divisible by an axisymmetric determinant. *Messenger of Math.*, xxxvii. pp. 42–48.]

[*Nouv. Annales de Math.*, (4) ix. pp. 175–177.]

The 'simultaneous vanishing' was what occurred to Sylvester, and his paper of 1853 (*Hist.*, ii. pp. 127–129) tells what came of the thought. The advance here made is in two directions, the

* The following 'questions' are passed over, having already received due attention: Question 578 in *Math. Gazette*, iv. pp. 60–61: Questions 153, 303 in *American Math. Monthly*, xv. pp. 144, 169, xvi. pp. 11–12.

problem being freed from geometrical restrictions, and the algebraical theorem being generalized. The latter stands as follows: *If* [rr] *be the cofactor of the element* (rr) *in the axisymmetric determinant* Δ *of the* nth *order, then the norm of*

$$([11]^{\frac{1}{2}}, [22]^{\frac{1}{2}}, \ldots, [nn]^{\frac{1}{2}} \between \text{any row of } \Delta)$$

is divisible by Δ. For example, the determinant and its adjugate being

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}, \quad \begin{vmatrix} A & B & C \\ B & D & E \\ C & E & F \end{vmatrix}$$

the norm of $a\sqrt{A} + b\sqrt{D} + c\sqrt{F}$

$$= a^2 \begin{vmatrix} 2 & a & d & f \\ a & . & b^2 & c^2 \\ d & b^2 & . & A \\ f & c^2 & A & . \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} \cdot a^2 \begin{vmatrix} a & b & -c \\ b & d & e \\ -c & e & f \end{vmatrix}.$$

Sylvester's is the very special case where the determinant is of the fifth order, the diagonal elements are zeros, and the elements of its last row and last column are units. The theorem being established a number of additional special cases are next considered, the object being, as with Sylvester, to determine the cofactor of Δ in the norm.

The interest of the second paper lies in the alternative proof given of the same theorem, this being now very simply based on the well-known property of an axisymmetric determinant that has a vanishing coaxial minor. Space is also devoted to some other special instances of the factorization of the norm.

The remaining paper merely gives an account of Muir's work.

DUBOUIS, E. (1908$^1/_4$): NANSON, E. J. (1910$^1/_3$)

[Question 3378. *L'Intermédiaire des Math.*, xv. pp. 97, 256.]

[Question 16830. *Educ. Times*, lxiii. p. 135: or *Math. from Educ. Times*, (2) xx. p. 108.]

The only interesting point here is connected with the second of the questions, namely, the appearance of the axisymmetric form of Sylvester's unisignant of 1855 as the discriminant of a quadric, for example, of the quadric

$$a(x-y)^2+b(x-z)^2+c(x-w)^2+d(y-z)^2+e(y-w)^2+f(z-w)^2.$$

(See *Hist.*, ii. pp. 456–459.)

MUIR, T. ($1909^{15}/_4$)

[Borchardt's form of the eliminant of two equations of the n^{th} degree. *Trans. R. Soc. S. Africa*, i. pp. 447–452.]

The subject of the second part of this paper is the axisymmetric case of Sylvester's unisignant of 1885 (*Hist.*, ii. pp. 456–459). That a function of six variables $a_2, a_3, a_4, b_3, b_4, c_4$, which in non-determinant notation appears as the sum of 16 positive terms, namely,

$$a_2a_3a_4 + a_2a_3(b_4+c_4) + a_2a_4(b_3+c_4) + a_3a_4(b_3+b_4) \\ + (a_2+a_3+a_4)(b_3b_4+b_3c_4+b_4c_4)$$

should be represented by

$$\begin{vmatrix} a_2+b_3+b_4 & -b_3 & -b_4 \\ -b_3 & a_3+b_3+c_4 & -c_4 \\ -b_4 & -c_4 & a_4+b_4+c_4 \end{vmatrix}$$

the expansion of which has 38 terms, 22 of which cancel each other, seems to the author interesting but on practical grounds indefensible. As a substitute he uses a Pfaffian-like notation whose recurrent law of formation is quite simple and whose development is hampered by no superfluous terms; for example,

$$\{a_2\} \equiv a_2, \quad \left\{\begin{matrix} a_2 & a_3 \\ & b_3 \end{matrix}\right\} \equiv (a_2+a_3)\{b_3\} + a_2a_3,$$

$$\left\{\begin{matrix} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{matrix}\right\} \equiv (a_2+a_3+a_4)\left\{\begin{matrix} b_3 & b_4 \\ & c_4 \end{matrix}\right\} + a_2a_3\{b_4+c_4\} \\ + a_2a_4\{b_3+c_4\} + a_3a_4\{b_3+b_4\} + a_2a_3a_4,$$

$$\left\{\begin{matrix} a_2 & a_3 & a_4 & a_5 \\ & b_3 & b_4 & b_5 \\ & & c_4 & c_5 \\ & & & d_5 \end{matrix}\right| \equiv (a_2+a_3+a_4+a_5)\left\{\begin{matrix} b_3 & b_4 & b_5 \\ & c_4 & c_5 \\ & & d_5 \end{matrix}\right\}$$
$$+\Sigma a_2 a_3 \left\{\begin{matrix} b_4+c_4 & b_5+c_5 \\ & d_5 \end{matrix}\right\}$$
$$+\Sigma\, a_2 a_3 a_4 \{b_5+c_5+d_5\} + a_2 a_3 a_4 a_5.$$

It is pointed out that with the elements in the umbral notation the law is still more readily apparent, and the properties of the functions more readily discussed.

NICOLETTI, O. (1909²/₅): ESCOTT, E. B. (1910¹/₅)

[Question 16863. *Educ. Times*, lxiii. pp. 210, 462: or *Math. from Educ. Times*, (2) xix. p. 90.]

The equality here dealt with by Escott is best viewed as a case of

$$\begin{vmatrix} (N)_0 & (N+1)_1 & (N+2)_2 & \dots \\ (N+1)_0 & (N+2)_1 & (N+3)_2 & \dots \\ (N+2)_0 & (N+3)_1 & (N+4)_2 & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 1,$$

namely, the case where $N = 0$ (*Hist.*, iii. p. 447).

Nicoletti's paper of the date mentioned is reported on in Chapter XI.

MUIR, T. (1910¹²/₁₂)

[A new unisignant. *Messenger of Math.*, xl. pp. 177–192.]

This is another unisignant of some rather special interest, which like Borchardt's and Boole's is axisymmetric in form. It is a function of as many variables as it has coaxial minors, one variable being associated with each coaxial minor through forming a separate part of every element of the latter. It thus may also be described as a determinant with polynomial elements such that all the elements of each coaxial minor have a term in

common. When the determinant is of the third order there are seven (i.e. $2^3 - 1$) coaxial minors, namely:

the determinant itself $|\,1_1\ 2_2\ 3_3\,|$,
the three 2-line minors $|\,2_2\ 3_3\,|$, $|\,1_1\ 3_3\,|$, $|\,1_1\ 2_2\,|$,
and the three 1-line minors 1_1, 2_2, 3_3;

so that if the variables corresponding to these be respectively

$$\begin{array}{ccc} a & & \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{array}$$

the determinant is

$$\begin{vmatrix} a+b_2+b_3+c_1 & a+b_3 & a+b_2 \\ a+b_3 & a+b_1+b_3+c_2 & a+b_1 \\ a+b_2 & a+b_1 & a+b_1+b_2+c_3 \end{vmatrix}.$$

In investigating its properties and in leading up to evaluation a convenient notation is found to be

$$M(a;\ b_1, b_2, b_3;\ c_1, c_2, c_3), \text{ or } M_3:$$

for example,

$$\begin{aligned} M_3 &= M(a;\ b_3, b_1, b_2;\ c_3, c_1, c_2) = \ldots \\ &= M(c_1;\ b_1, b_3, b_2;\ a, c_2, c_3) = \ldots \end{aligned}$$

Of the results obtained we need only note two, the seventh and thirteenth, namely: *A variable cannot occur in a higher power than the first in any term of the final development of* M, *and its cofactor is always obtainable in the form of an axisymmetric determinant whose order-number is one less than that of* M. *In an* M *of the* n*th* *order the number of terms is* $2^{n(n+1)} \cdot (n+1)$.

MUIR, T. ($1911^{17}/_1$)

[Boole's unisignant. *Proceed. R. Soc. Edinburgh*, xxxi. pp. 448–455.]

The need, which we have already referred to (*Hist.*, iii. pp. 98–100), for an alternative proof of Boole's general proposition is here supplied; and, fortunately, it is not only brief but usefully

corrective and supplementary. His less general determinant, which for the 4th order and

$$V = axyz + (b, c, d \between yz, zx, xy) + (e, f, g \between x, y, z) + h,$$

is

$$\begin{vmatrix} V & x\dfrac{\partial V}{\partial x} & y\dfrac{\partial V}{\partial y} & z\dfrac{\partial V}{\partial z} \\ x\dfrac{\partial V}{\partial x} & x\dfrac{\partial V}{\partial x} & xy\dfrac{\partial^2 V}{\partial x \partial y} & xz\dfrac{\partial^2 V}{\partial x \partial z} \\ y\dfrac{\partial V}{\partial y} & xy\dfrac{\partial^2 V}{\partial x \partial y} & y\dfrac{\partial V}{\partial y} & yz\dfrac{\partial^2 V}{\partial y \partial z} \\ z\dfrac{\partial V}{\partial z} & xz\dfrac{\partial^2 V}{\partial x \partial z} & yz\dfrac{\partial^2 V}{\partial y \partial z} & z\dfrac{\partial V}{\partial z} \end{vmatrix}$$

is next investigated at length, and a series of results obtained, including several forms of expression for the final expansion. This work is facilitated by the use of a notation similar to that of the author's immediately preceding paper, e.g. the just-mentioned 4-line determinant is denoted by

$$B(a;\ b, c, d;\ e, f, g;\ h) \quad \text{or} \quad B_4.$$

Of the results perhaps the most noteworthy is the generalization: *Any positive elements whatever may be inserted in the vacant places of the determinant*

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ -\alpha & -\alpha & \cdot & \cdot \\ -\beta & \cdot & -\beta & \cdot \\ -\gamma & \cdot & \cdot & -\gamma \end{vmatrix}$$

and yet all the terms of the final development remain positive; further, if the elements so inserted be $\alpha_3, \alpha_2, \beta_3, \beta_1, \gamma_2, \gamma_1$, *the three-line determinant*

$$\begin{vmatrix} \alpha + \alpha_2 + \alpha_3 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta + \beta_3 + \beta_1 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma + \gamma_1 + \gamma_2 \end{vmatrix}$$

has the same development, namely,

$$\alpha\beta\gamma + \alpha\beta(\gamma_1 + \gamma_2) + \beta\gamma(\alpha_2 + \alpha_3) + \gamma\alpha(\beta_3 + \beta_1)$$
$$+ \alpha(\beta_3\gamma_1 + \beta_1\gamma_1 + \beta_1\gamma_2) + \beta(\alpha_2\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_2)$$
$$+ \gamma(\alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_3\beta_3) + 2(\alpha_2\beta_3\gamma_1 + \alpha_3\beta_1\gamma_2).$$

We may also note: *When* B *is of the* n^{th} *order, the number of variables is* 2^{n-1}, *and the number of terms in the final development is*

$$2^{(n-1)(n-2)}.$$

MUIR, T. (1911 25/1)

[Cayley's linear relation between minors of a special three-row array. *Messenger of Math.*, xli. pp. 23–28.]

The linear relation in question had hitherto been viewed as a subject by itself (*Hist.*, ii. pp. 465–467). Now when subjected to careful scrutiny it turns out to be classifiable as a vanishing aggregate of 4-line determinants, being when n is 4 included in

$$\begin{vmatrix} \theta_2 & \theta_3 & \theta_1 & \theta_4 \\ a & b & c & d \\ b & c & D & e \\ \alpha & \beta & \gamma & \delta \end{vmatrix} + \begin{vmatrix} a & b & d & \theta_1 \\ a & b & d & c' \\ b & c & e & d' \\ \alpha & \beta & \delta & c'' \end{vmatrix} + \begin{vmatrix} a & b & \omega & \theta_2 \\ b & c & d & b' \\ c & D & e & c' \\ \beta & \gamma & \delta & b'' \end{vmatrix} + \begin{vmatrix} b & D & e & \theta_3 \\ a & c & d & a' \\ b & D & e & b' \\ \alpha & \gamma & \delta & a'' \end{vmatrix} + \begin{vmatrix} \omega & d & e & \theta_4 \\ a & b & c & b' \\ b & c & D & e' \\ \alpha & \beta & \gamma & b'' \end{vmatrix} = 0$$

and when n is 5 in

$$\begin{vmatrix} a & b & m & \theta_1 \\ a & b & m & v \\ b & e & n & y \\ c & f & o & z \end{vmatrix} + \begin{vmatrix} a & b & \omega_1 & \theta_2 \\ b & e & m & s \\ e & h & n & v \\ f & i & o & w \end{vmatrix} + \begin{vmatrix} a & b & \omega_2 & \theta_3 \\ e & h & m & q \\ h & k & n & s \\ i & l & o & t \end{vmatrix} - \begin{vmatrix} b & k & n & \theta_6 \\ a & h & m & p \\ b & k & n & q \\ c & l & o & r \end{vmatrix}$$

$$- \begin{vmatrix} \omega_2 & m & n & \theta_5 \\ a & e & h & q \\ b & h & k & s \\ c & i & l & t \end{vmatrix} - \begin{vmatrix} \omega_1 & m & n & \theta_4 \\ a & b & e & s \\ b & e & h & v \\ c & f & i & w \end{vmatrix} + \begin{vmatrix} \theta_2 & \xi & \theta_1 & \theta_4 \\ a & b & e & m \\ b & e & h & n \\ c & f & i & o \end{vmatrix} + \begin{vmatrix} \theta_3 & \theta_6 & \xi & \theta_5 \\ a & e & h & m \\ b & h & k & n \\ c & i & l & o \end{vmatrix} = 0.$$

Involved as the cofactor of s in the latter is the equality

$$\begin{vmatrix} a & b & \omega_1 \\ e & h & n \\ f & i & o \end{vmatrix} - \begin{vmatrix} a & b & \omega_2 \\ e & h & m \\ i & l & o \end{vmatrix} + \begin{vmatrix} \omega_2 & m & n \\ a & e & h \\ c & i & l \end{vmatrix} - \begin{vmatrix} \omega_1 & m & n \\ b & e & h \\ c & f & i \end{vmatrix} = 0,$$

which, though outwardly resembling Kronecker's, is quite distinct from it. As regards the new aggregate it is noted that when the number of columns in the Cayleyan array is $2n-1$, the number of variables involved is $8n-10$ and the number of determinants is $3n-7$.

The paper closes with a curious restatement of the basic fact in Kronecker's relation, namely, *If an array of* n *rows and* 2n − 2 *columns have its first principal minor symmetric with respect to a zero diagonal, the aggregate of the* (n − 1)*-line minors which are free of zero elements vanishes.* Thus, when n is 4 such an array is

$$\begin{array}{cccccc} \cdot & a & b & c & \alpha_1 & \alpha_2 \\ a & \cdot & d & e & \beta_1 & \beta_2 \\ b & d & \cdot & f & \gamma_1 & \gamma_2 \\ c & e & f & \cdot & \delta_1 & \delta_2 \end{array}$$

and we have

$$\begin{vmatrix} a & \beta_1 & \beta_2 \\ b & \gamma_1 & \gamma_2 \\ c & \delta_1 & \delta_2 \end{vmatrix} - \begin{vmatrix} a & \alpha_1 & \alpha_2 \\ d & \gamma_1 & \gamma_2 \\ e & \delta_1 & \delta_2 \end{vmatrix} + \begin{vmatrix} b & \alpha_1 & \alpha_2 \\ d & \beta_1 & \beta_2 \\ f & \delta_1 & \delta_2 \end{vmatrix} - \begin{vmatrix} c & \alpha_1 & \alpha_2 \\ e & \beta_1 & \beta_2 \\ f & \gamma_1 & \gamma_2 \end{vmatrix} = 0.$$

KAPADIA, D. D. (1911/$_4$)

[Question 288. *Journ. Indian Math. Soc.*, iii. p. 90, pp. 200–201.]

A determinant closely related to Walker's penesymmetric of 1870 (*Hist.*, iii. p. 127).

MUIR, T. (1911^{24}/$_5$)

[Sylvester's axisymmetric unisignant. *Transac. R. Soc. S. Africa*, ii. pp. 197–202.]

In effect this is a continuation of the author's investigation of 1909^{15}/$_4$ (see above, p. 147). An alternative law of development is now found, illustrated for the case of $\mathfrak{S}_3$ by

$$\left|\begin{Bmatrix} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{Bmatrix}\right| = a_2\left|\begin{Bmatrix} b_3+a_3 & b_4+a_4 \\ & c_4 \end{Bmatrix}\right| + a_3\left|\begin{Bmatrix} b_3 & b_4 \\ & c_4+a_4 \end{Bmatrix}\right| + a_4\left|\begin{Bmatrix} b_3 & b_4 \\ & c_4 \end{Bmatrix}\right|,$$

and is successfully applied to find the final development. The adjugate of $\mathfrak{S}$ is next considered. Being an integral power of $\mathfrak{S}$ it also must, of course, be a unisignant: one is not prepared, however, for the interesting theorem that it belongs to the M type. This is established, and the actual expressions of the adjugates of $\mathfrak{S}_3$, $\mathfrak{S}_4$ as M's of the 3rd and 4th order are given. Proof is also adduced that the primary minors of $\mathfrak{S}$ and the determinant formed by bordering $\mathfrak{S}$ axisymmetrically by 0, x, y, z, ... are all of them unisignant.*

MUIR, T. ($1911^2/_{10}$)

[Lagrange's determinantal equation in the case of a circulant. *Messenger of Math.*, xli. pp. 167–174.]

In the course of this paper we unexpectedly learn that there is a family of axisymmetric determinants which are resolvable into linear factors closely after the manner of circulants. For example,

$$\begin{vmatrix} a_1 - a_2 & a_2 - a_3 & a_3 - a_4 \\ a_2 - a_3 & a_1 - a_4 & a_2 - a_4 \\ a_3 - a_4 & a_2 - a_4 & a_1 - a_3 \end{vmatrix}$$

$$= \left[\begin{array}{l} \{a_1 + (\eta + \eta^6)a_2 + (\eta^2 + \eta^5)a_3 + (\eta^3 + \eta^4)a_4\} \\ \cdot\{a_1 + (\eta^2 + \eta^5)a_2 + (\eta^4 + \eta^3)a_3 + (\eta^6 + \eta)a_4\} \\ \cdot\{a_1 + (\eta^3 + \eta^4)a_2 + (\eta^6 + \eta)a_3 + (\eta^2 + \eta^5)a_4\} \end{array}\right.$$

where η is a primitive 7th root of 1. The general theorem stands thus: *The axisymmetric determinant which has the differences* $a_1 - a_2, a_1 - a_3, \ldots, a_1 - a_n$ *in the* 1*st*, n*th*, 2*nd*, (n − 1)*th*, ... *places of the principal diagonal, the differences* $a_2 - a_3, a_2 - a_4$, ... *similarly disposed in the adjacent minor diagonal, the differences* $a_3 - a_4, a_3 - a_5$, ... *similarly disposed in the next diagonal, and so on, is resolvable into linear factors, being equal to the* n − 1 *different expressions of the form*

$$a_1 + (\omega + \omega^{2n-2})a_2 + (\omega^2 + \omega^{2n-3})a_3 + \ldots + (\omega^{n-1} + \omega^n)a_n$$

* Unisignants that are not axisymmetric are dealt with in Chap. XI.

where ω *is a primitive* $(2n-1)^{th}$ *root of* 1. For the purposes of proof it is sufficient to note the first operation

$$\mathrm{col}_1 + (1+\theta_1)\,\mathrm{col}_2 + (1+\theta_1+\theta_2)\,\mathrm{col}_3 + \ldots$$

where θ_r stands for $\omega^r + \omega^{rn-n-1}$.

ATKIN, A. L. (1912$^1/_2$)

[Question 17238. *Educ. Times*, lxv. p. 81.]

The proposer here draws attention to the rules given in an appendix to § 280 of Maxwell's *Electricity and Magnetism* * for the evaluation of the cofactor of K_{nn} in the determinant $|K_{11}K_{22}\ldots K_{nn}|$ whose row-sums all vanish. Unfortunately he elicited nothing; an addition would have been welcome to what had been done on the subject by Sylvester in 1855 and Borchardt in 1859 (*Hist.*, ii. pp. 150, 406–7, 456–9).

SWAMINARAYAN, J. C. (1912/$_6$)

[The volume of a tetrahedron in terms of its six edges. *Journ. Indian Math. Soc.*, iv. pp. 56–58.]

[Question 377. *Journ. Indian Math. Soc.*, iv. p. 118, pp. 195–197.]

In the first of these contributions allied matters (*Hist.*, ii. pp. 109–110, . . .) are usefully brought in and dealt with: the second concerns the axisymmetric determinant got by annexing a 2-line border to a 3-line axisymmetric determinant (*Hist.*, iii. pp. 435–436, . . .).

MUIR, T. (1912$^{24}/_6$)

[The theory of axisymmetric determinants from 1857 to 1880. *Proceed. R. Soc. Edinburgh*, xxxiii. pp. 49–63.]

This, the third of our reports on the history of axisymmetric determinants, contains an account of about two dozen writings

* The first edition of this was published in 1873: § 280 there occupies pp. 333–335 of Vol. I.

of varied interest, the average value of them being certainly lower than that of the previous report.

BOTTASSO, M. ($1913^{16}/_1$)

[Sui sistemi di equazioni ottenuti da un determinante simmetrico di forme in più serie di variabili. *Rendic. del R. Istituto Lombardo. . . .*, (2) xlvi. pp. 88–103.]

The title of this paper is appropriate only to the second half * of it: the other half, as we have already partly indicated in our first chapter, is occupied in the establishment of preparatory theorems that bear directly on determinants. Unfortunately those of them whose basic determinant is axisymmetric do not permit of very simple enunciation, a fact indeed which weighs so much on the author that he is driven to adopt a novel expedient to secure condensation. This consists in saying that 'the condition $S(m, c, \mu)$ is satisfied' when he means that

the determinant $\begin{vmatrix} 0, 1, \ldots, m \\ 0, 1, \ldots, m \end{vmatrix}$ is axisymmetric,

all its $(c + 1)$-line minors vanish,

and $\left\| \begin{matrix} 0, 1, \ldots, \mu \\ 0, 1, \ldots\ldots, m \end{matrix} \right\| \equiv 0$ where $\mu < c$.

For example, the first and simplest theorem is: *If* S(m, c, μ) *be satisfied and* $\begin{vmatrix} 0, 1, \ldots, \mu - 1 \\ 1, 2, \ldots, \mu \end{vmatrix} = 0$, *then either* S(m, c, $\mu - 1$) *is satisfied or* $\left\| \begin{matrix} 0, 1, \ldots, \mu \\ 1, \ldots, \mu \end{matrix} \right\| \equiv 0.$

DICKSON, L. E. ($1913^{22}/_3$): WEDDERBURN, J. H. M. ($1913/_9$)

[On the rank of a symmetrical matrix. *Annals of Math.*, xv. pp. 27–28.]

[Note on the rank of a symmetrical matrix. *Annals of Math.*, xv. p. 29.]

* Which half is classifiable along with Giambelli's papers in the Auxiliary List of Chap. I.

The object of the opening paper here is to give a proof of the theorem—called, without assigned reason, Kronecker's—that *if in an axisymmetric determinant the minors of order* r + 1 *all vanish but not all those of order* r, *then one of the latter must be coaxial.* With us this has twice appeared under Frobenius, first in 1876 (*Hist.*, iii. pp. 275–277) and again in 1894 (*Hist.*, iv. p. 135). Whoever the original author may be, it is clear that these two papers of Frobenius' must now have been overlooked, because in them is also found formulated the lemma on which the proof here offered rests.

The second communication is on fresher lines, the determinant in it being viewed as the discriminant of a quadric, and the reasoning thereby becoming partly geometrical. The desired result is reached by first showing anew that the sum of the r-line coaxial minors is not 0.

MUIR, T. ($1913^1/_{10}$)

[Question 17601. *Educ. Times*, lxvi. p. 434; lxvii. pp. 204–205: or *Math. from Educ. Times*, (2) xxvi. pp. 69–71.]

The curious result here drawn attention to is that the axisymmetric determinant

$$\begin{vmatrix} ax - by - cz & ay + bx & az + cx \\ ay + bx & by - cz - ax & bz + cy \\ az + cx & bz + cy & cz - ax - by \end{vmatrix}$$

is equal to the skew determinant

$$\begin{vmatrix} ax + by + cz & ay - bx & az - cx \\ bx - ay & ax + by + cz & bz - cy \\ cx - az & cy - bz & ax + by + cz \end{vmatrix}$$

and that the like equality holds for the n^{th} order, the common value then being $(\Sigma ax)^{n-2}\,\Sigma a^2 \Sigma x^2$.

WILKINSON, A. C. L. ($1913/_{10}$)

[Question 495. *Journ. Indian Math. Soc.*, v. p. 100; viii. p. 142.]*

The interesting result here brought to notice is

$$\begin{vmatrix} 1 & \cos c & \cos b & \cos(b-c) \\ \cos c & 1 & \cos a & \cos(c-a) \\ \cos b & \cos a & 1 & \cos(a-b) \\ \cos(b-c) & \cos(c-a) & \cos(a-b) & 1 \end{vmatrix}$$

$$= -16\sin^2\tfrac{1}{2}(b+c-a)\sin^2\tfrac{1}{2}(c+a-b)\sin^2\tfrac{1}{2}(a+b-c).$$

It is included in the fact that

$$\begin{vmatrix} 1 & lm-pq & nl-rp & mn+qr \\ lm-pq & 1 & mn-qr & nl+rp \\ nl-rp & mn-qr & 1 & lm+pq \\ mn+qr & nl+rp & lm+pq & 1 \end{vmatrix} = -16p^2q^2r^2$$

when $1 = l^2+p^2 = m^2+q^2 = n^2+r^2$. A proof is supplied by N. S. Aiyar, but something less laborious is a thing to be desired.

METZLER, W. H. ($1913/_{11}$)

[On the rank of a matrix. *Annals of Math.*, xv. pp. 161–165.]

Only a page of this paper is at present of interest to us, the essential part being the conclusion, namely, that *if in an axisymmetric determinant all the coaxial minors of order* m + h *and* m + h + 1 *that 'contain' the coaxial minor* M *of order* m *be zero, then all the other minors of order* m + h *that 'contain'* M *are zero likewise.*

* The following 'questions' have already received due attention: Question 17546 in *Educ. Times*, lxvi. p. 302: Question 17930 in *Math. Quest. and Sol.*, i. p. 27, ii. p. 59: Question 18016 in *Math. from Educ. Times*, (2), xxix. pp. 102–103: Question 433 in *American Math. Monthly*, xxii. p. 161: Question 18250 in *Math. Quest. and Sol.*, iii. pp. 8–9.

MUIR, T. ($1914^{20}/_{2}$)

(See under this heading in Chapter I and Chapter X.)

BLISS, G. A. ($1914/_{9}$): WEDDERBURN, J. H. M. ($1914/_{12}$)

[A note on symmetric matrices. *Annals of Math.*, xvi. pp. 43–44.]
[Note on the rank of a symmetrical matrix. *Annals of Math.*, xvi. pp. 86–88.]

The subject here is again the establishment of Frobenius' first corollary of 1876.* One cannot complain of over-demonstration, as both the new contributions are instructive and interesting. The one writer finds a foundation in the lemma that *in the circumstances under consideration there exists a non-zero multiplier* M *and* $(n)_r$ *quantities* $X_1, X_2, \ldots$, *not all zero, such that the* $(ij)^{th}$ *element of the determinant of the* r-*line minors equals* MX_iX_j, this in its turn being dependent on Frobenius' first theorem of all regarding vanishing minors (*Hist.*, iii. p. 63). The other writer extends his former note so as not to leave unconsidered the so-called case where the original elements are complex.

MUIR, T. ($1914^{1}/_{10}$)

[Question 17833. *Educ. Times*, lxvii. p. 479: *Math. from Educ. Times*, (2) xxviii. pp. 42–43.]

The equality here noted is

$$\begin{vmatrix} a & x_1x_2 & x_1x_3 & x_2x_3 \\ x_1x_2 & b & x_1x_4 & x_2x_4 \\ x_1x_3 & x_1x_4 & c & x_3x_4 \\ x_2x_3 & x_2x_4 & x_3x_4 & d \end{vmatrix} = (a-\delta_4)(b-\delta_3)(c-\delta_2)(d-\delta_1) \times\left\{1+\frac{\delta_4}{a-\delta_4}+\frac{\delta_3}{b-\delta_3}+\frac{\delta_2}{c-\delta_2}+\frac{\delta_1}{d-\delta_1}\right\}$$

where $\delta_4 = x_1x_2x_3/x_4$, $\delta_3 = x_1x_2x_4/x_3, \ldots$

The determinant is of some little interest as being a companion to another already well known (*Hist.*, iii. pp. 125–126, 131), the one being changeable into Sardi's form by multiplication and the other by division.

* Wedderburn properly supplies the reference to Frobenius.

VYTHYNATHYSWAMY, R. (1914/$_{10}$): THRIVEDI, T. P. (1915/$_{6}$)

[Questions 580, 657. *Journ. Indian Math. Soc.*, vi. pp. 198–199; viii. pp. 63–66.]

The subject of the first of these is Sylvester's result of 1853 dealt with in Muir's papers of 1906, 1907 reported on above, and that of the second is Ferrers' of 1861 (*Hist.*, iii. pp. 97–98).

MUIR, T. (1915^1/$_1$)

[Question 17904. *Educ. Times*, lxviii. p. 38, pp. 384–385; or *Math. from Educ. Times*, (2) xxix. pp. 67–68.]*

What is here established is the curious equality

$$\begin{vmatrix} \cdot & a & b & c \\ a & 2d & d+e & d+f \\ b & d+e & 2e & e+f \\ c & d+f & e+f & 2f \end{vmatrix} = \begin{vmatrix} \cdot & a & b & c \\ a & -d+e+f & f & e \\ b & f & d-e+f & d \\ c & e & d & d+e-f \end{vmatrix};$$

also the fact that neither determinant is altered in substance by a, b, c being interchanged with d, e, f as each is equal to

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ d & e & f \end{vmatrix}^2.$$

The two 3-line determinants which are subjected to bordering are both axisymmetric, and both $= 0$.

More interesting is it to find a relation of equality between an axisymmetric and a Pfaffian, namely,

$$\begin{vmatrix} \cdot & (a_1-a_2)^4 & (a_1-a_3)^4 & (a_1-a_4)^4 \\ (a_1-a_2)^4 & \cdot & (a_2-a_3)^4 & (a_2-a_4)^4 \\ (a_1-a_3)^4 & (a_2-a_3)^4 & \cdot & (a_3-a_4)^4 \\ (a_1-a_4)^4 & (a_2-a_4)^4 & (a_3-a_4)^4 & \cdot \end{vmatrix}$$

$$= -\tfrac{16}{5}\zeta^{\frac{1}{2}} \cdot \left| \begin{array}{ccc} (a_1-a_2)^5 & (a_1-a_3)^5 & (a_1-a_4)^5 \\ & (a_2-a_3)^5 & (a_2-a_4)^5 \\ & & (a_3-a_4)^5 \end{array} \right|.$$

* The results given in Questions 17930, 18016 are already familiar to us. The same is true of Question 433 in the *American Math. Monthly*, xxii. p. 161.

MUIR, T. ($1916^{19}/_1$)

[The recurrence-formula of Jacobi's persymmetric determinant. *Tôhoku Math. Journ.*, ix. pp. 239–244.]

Incidentally it is here shown that *In any bordered axisymmetric determinant the difference of the complementary minors of any two conjugate elements is expressible as a single determinant:* for example, the difference of the two complementary minors of d_4 in

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_1 & b_1 & b_2 & b_3 & b_4 \\ y_2 & b_2 & c_2 & c_3 & c_4 \\ y_3 & b_3 & c_3 & d_3 & d_4 \\ y_4 & b_4 & c_4 & d_4 & e_4 \end{vmatrix} \quad \text{is} \quad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_2 & c_2 & c_3 & c_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}$$

SUTÔ, O. ($1916/_2$)

[Application of the notion of the *class* of a matrix in the theory of quadratic forms. *Tôhoku Math. Journ.*, ix. pp. 61–77.]

The interest here lies in the use of the notion referred to in the title. In a previous paper (vii. pp. 38–53) the author had laid down the definition: *The index of a vanishing array is said to be* v *if there be in it a set of* v *rows that vanishes identically and a set of* v − 1 *rows that does not.* In the present paper he departs from this and speaks instead of such an array being of the v^{th} *class,** the proposal being in loose analogy with Frobenius' use of the term ‘rank’. As to the possible usefulness of the new concept some idea may be formed from the enunciation of the main theorem calling for our attention in the paper: it is that *in an axisymmetric determinant there exist at least* v r-*line co-axial minors that do not vanish,* v and r being the so-called ‘class’ and ‘rank’ respectively.

* The matter concerned with being that with which the words ‘evanescent’ ‘evanescence’ have come to be associated, the term ‘index of evanescence’ would be less vague.

ONO, T. ($1916^1/_2$)

[Question 2286. *Nouv. Annales de Math.*, (4) xvi. p. 96: xviii. p. 265.]

Ono's result is an interesting companion to Muir's of the preceding year, namely,

$$\begin{vmatrix} 2abc & -c^3 & -b^3 & a \\ -c^3 & 2abc & -a^3 & b \\ -b^3 & -a^3 & 2abc & c \\ a & b & c & . \end{vmatrix} = \begin{vmatrix} 2a^2 & -c^2 & -b^2 & a^2 \\ -c^2 & 2b^2 & -a^2 & b^2 \\ -b^2 & -a^2 & 2c^2 & c^2 \\ a^2 & b^2 & c^2 & . \end{vmatrix}$$
$$= (a^2+b^2+c^2)^2(a+b+c)(a+b-c)(a-b+c)(a-b-c),$$

the operation $(bc-a^2)\text{col}_4+\text{col}_3+\text{col}_2+\text{col}_1$ performed on the first determinant giving the factor $a+b+c$, and the operation $-3\,\text{col}_4+\text{col}_3+\text{col}_2+\text{col}_1$ performed on the second giving the factor $a^2+b^2+c^2$.

We may add that the operations

$$\text{row}_1 - bc\,\text{row}_4,\quad \text{row}_2 - ca\,\text{row}_4,\quad \text{row}_3 - ab\,\text{row}_4;$$
$$\text{col}_1 - bc\,\text{col}_4,\quad \text{col}_2 - ca\,\text{col}_4,\quad \text{col}_3 - ab\,\text{col}_4;$$

enable us to remove the factor $(\Sigma a^2)^2$ and leave a determinant whose factors are well known (*Hist.*, ii. p. 134); and that the operations

$$\text{row}_1 - \text{row}_4,\quad \text{row}_2 - \text{row}_4,\quad \text{row}_3 - \text{row}_4$$
$$\text{col}_1 - \text{col}_4,\quad \text{col}_2 - \text{col}_4,\quad \text{col}_3 - \text{col}_4$$

performed on the second have the like effect.

The three originals are connected by the factor abc, and hence the difference in their borders.

ALLER, C. VAN (1916): DATTA, H. ($1916^9/_6$)

[Vraagstukken 35, 36. *Wiskundige Opgaven*, xii. pp. 77–87.]

[On symmetric determinants and Pfaffians. *Proceed. Edinburgh Math. Soc.*, xxxiv. pp. 197–204.]

The two results, very fully dealt with in the first of these contributions, are

$$\left| \sin\frac{1\cdot1\pi}{n} \cdot \sin\frac{2\cdot2\pi}{n} \ldots \sin\frac{(n-1)^2\pi}{n} \right| = (-1)^{\frac{1}{2}(n-1)(n-2)} \cdot (\tfrac{1}{2}n)^{\frac{1}{2}(n-1)},$$

$$\left| \cos\frac{1\cdot1\pi}{n} \cdot \cos\frac{2\cdot2\pi}{n} \ldots \cos\frac{(n-1)^2\pi}{n} \right| = (-1)^{\frac{1}{2}n(n-1)} \cdot (\tfrac{1}{2}n)^{\frac{1}{2}(n-1)}.$$

The former is Muir's of 1882 (*Hist.*, iv. pp. 111–112), and the latter is of the same type as Hunyady's of 1872 (*Hist.*, iii. p. 104); but it has also to be noted that along with his proofs of these a contributor, J. G. v. d. Corput, furnishes an extensive list of analogous results.

The determinants referred to in the second contribution are persymmetric, and are therefore more conveniently dealt with elsewhere.

ROSS, C. M. (1916¹/₁₁): ANNING, N. (1917¹/₁₀): ROSS, C. M. (1917¹/₁₁)

[Question 18320. *Math. Quest. and Sol.*, iii. pp. 26–27.]

[Question 488. *American Math. Monthly*, xxiv. p. 388.]

[Question 18537. *Math. Quest. and Sol.*, v. pp. 65–66.]

The first determinant here is the result of bordering the persymmetric determinant of $\cos x$, $\cos 2x$, $\cos 3x$, ... axisymmetrically with the line

$$-\cos x, \sin x, \sin 2x, \sin 3x, \ldots$$

It is curious to note that it and the persymmetric determinant are equal to

$$\left\| \begin{matrix} 0 & -1 \\ 1 & 0 \\ \cos x & \sin x \\ \cos 2x & \sin 2x \\ \cdot\ \cdot & \cdot\ \cdot \end{matrix} \right\| \cdot \left\| \begin{matrix} \sin x & \cos x \\ \cos x & -\sin x \\ \cos 2x & -\sin 2x \\ \cos 3x & -\sin 3x \\ \cdot\ \cdot & \cdot\ \cdot \end{matrix} \right\|,$$

$$\left\| \begin{matrix} 1 & 0 \\ \cos x & \sin x \\ \cos 2x & \sin 2x \\ \cdot\ \cdot & \cdot\ \cdot \end{matrix} \right\| \cdot \left\| \begin{matrix} \cos x & -\sin x \\ \cos 2x & -\sin 2x \\ \cos 3x & -\sin 3x \\ \cdot\ \cdot & \cdot\ \cdot \end{matrix} \right\|$$

respectively, and that therefore both vanish for any order higher than the second.

The second determinant, which also vanishes, is

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & \dfrac{1}{a_4} & 1 \\ a_2 & 1 & a_3 \end{vmatrix} \quad \text{where } a_k = \frac{\sin(k\theta + \alpha)}{\sin k\theta}.$$

The third determinant, shown by Nanson and the proposer to be equal to $2(a_1{}^2 + a_2{}^2 + \ \ldots + a_n{}^2)^n$, or $2S^n$ say, is formed from $|\, a_r a_s \,|_n$ by augmenting each diagonal by S. It may conveniently be classed with a group exploited by Scott in 1879 (*Hist.*, iii. pp. 130–131).

MUIR, T. (1917¹/₈)

[Question 18484. *Math. Quest. and Sol.*, iv. pp. 82–83.]

It is here shown that A, B, C *being coaxial primary minors of any 4-line axisymmetric determinant* Δ, *it is possible to express each of the other primary minors in the form*

$$uA + vB + wC$$

where u, v, w *are functions of the elements of* Δ. The material used is a selection from the twelve vanishing expressions got by multiplying a row of the determinant by a row of the adjugate.

WEILL, M. (1918¹/₄)

[Théorèmes généraux sur des systèmes de courbes et de points. *Nouv. Annales de Math.*, (4) xviii. pp. 121–138.]

Determinants are here used in connection with the mutual distances of points, but the interest of the paper is almost entirely geometrical.

MUIR, T. (1918¹/₅)

[Question 18646. *Math. Quest. and Sol.*, vi. pp. 34–35.]

The result here is that if A, B, C, D, E, F be the determinants of any 2-by-4 array, then

$$-\frac{1}{2}\begin{vmatrix} . & \alpha & \beta & \gamma & \delta \\ \alpha & . & A^2 & B^2 & C^2 \\ \beta & A^2 & . & D^2 & E^2 \\ \gamma & B^2 & D^2 & . & F^2 \\ \delta & C^2 & E^2 & F^2 & . \end{vmatrix}$$

$$= \{\alpha DEF - \beta BCF + \gamma ACE - \delta ABD\}^2.$$

It is based on the fact that the cofactor of the $(1, 1)^{\text{th}}$ element is equal to 0.

In the case of the 2-by-6 array the determinants $A, B, \ldots$ of the array appear in the *fourth* power as elements of the bordered determinant.

ROBINSON, L. H. (1918⁴/₉)

[A curious system of polynomials. *Bull. American Math. Soc.*, xxv. pp. 51–52.]

The brevity of the abstract given of this paper probably accounts for its unattractiveness. It really concerns the interesting subject of sets of quadrics with a common discriminant. This will at once be seen if we change the author's 3-line example into Cayley's form

$$\left.\begin{array}{l} (\Delta \between x_1, y_1, z_1)^2 \\ (\Delta \between x_1, y_1, z_1 \between x_2, y_2, z_2) \\ (\Delta \between x_2, y_2, z_2)^2 \end{array}\right\} \text{ where } \Delta = \begin{vmatrix} a & b & c \\ b & f & g \\ c & g & h \end{vmatrix}.$$

The particular result touched on is a consequence of the vanishing of Δ.

SADANAND, . . (1919/$_{11}$)

[Question 1077. *Journ. Indian Math. Soc.*, xi. p. 200.]

An incorrect result that nevertheless awakens interest in regard to the determinant

$$\begin{vmatrix} . & a_1 & a_5 & a_4 & \alpha & 1 \\ a_1 & . & a_2 & a_6 & \beta & 1 \\ a_5 & a_2 & . & a_3 & \gamma & 1 \\ a_4 & a_6 & a_3 & . & \delta & 1 \\ \alpha & \beta & \gamma & \delta & . & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

and the cofactors of its five zero elements.

CHAPTER V

Symmetric Determinants that are not Axisymmetric from 1893 to 1919

Even the notable rate of increase reported in the introduction to the preceding chapter is exceeded here, the number of writings being more than double the number for the period 1884–1897. As before the two special types of determinant that make their appearance are the *penesymmetric* which originated with M. Roberts in 1864 and assumed its standard form with Sardi in 1868 (*Hist.*, iii. pp. 100–101, 125–126), and the *centrosymmetric* which with its pair of determinant factors was produced by Zehfuss in 1862 (*Hist.*, iii. p. 124).

In reporting a special type of penesymmetric evaluated by Wolstenholme in 1870 (*Hist.*, iii. pp. 126–127) we drew attention in a footnote to an alternative result: but unfortunately we omitted to refer to an important consequence to which we had thus been led, namely, that *the said type of penesymmetric is expressible as the product of two penesymmetrics*; for example,

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} \cdot \begin{vmatrix} a+b+\frac{1}{2}c & -\frac{1}{2}c & -\frac{1}{2}c \\ -\frac{1}{2}a & b+c+\frac{1}{2}a & -\frac{1}{2}a \\ -\frac{1}{2}b & -\frac{1}{2}b & c+a+\frac{1}{2}b \end{vmatrix}$$

$$= \begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}.$$

This is a direct deduction from the multiplication-theorem if the operation be performed in a row-by-column fashion. Associated with the result and verifying it is the evaluation of each of the three penesymmetrics involved. (See Muir's textbook of 1882, Ex. 4, 23, 25, 26 on pp. 106, 109.) *

*** Other penesymmetrics will be found evaluated on pp. 53, 67, 110.**

SCHOUTE, P. H. (1893)

[Vraagstuk 138. *Wiskundige Opgaven*, v. pp. 313–315.]

The determinant here is not essentially different from M. Roberts' of 1864 (*Hist.*, iii. pp. 100–101).

MUIR, T. ($1900/_3$)

[On certain aggregates of determinant minors. *Proceed. R. Soc. Edinburgh*, xxiii. pp. 142–154.]

From a general result obtained here (§ 10) there is deduced an amendment of the author's theorem of 1888 (*Hist.*, iv. p. 140), in which the real condition is seen to be not that the $2m$-line determinant should be wholly centrosymmetric, but merely that

$$\begin{vmatrix} 1 & 2 & \ldots & m \\ m+1 & m+2 & \ldots & 2m \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 2m & \ldots & m+2 & m+1 \\ m & \ldots & 2 & 1 \end{vmatrix}$$

should be identical element-by-element.

MUIR, T. ($1900^3/_{12}$)

[A peculiar set of linear equations. *Proceed. R. Soc. Edinburgh* xxiii. pp. 248–260.]

The set of equations in question is

$$\left.\begin{array}{cccccccc} x_1) & + & a_2x_2 & + & a_3x_3 & + \ldots + & a_nx_n & = 0 \\ {}_1 & + & (a_2+x_2) & + & a_3x_3 & + \ldots + & a_nx_n & = 0 \\ {}_1x_1 & + & a_2x_2 & + & (a_3+x_3) & + \ldots + & a_nx_n & = 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1x_1 & + & a_2x_2 & + & a_3x_3 & + \ldots + & (a_n+x_n) & = 0 \end{array}\right\}$$

one peculiarity of it being that any x and the corresponding a may be interchanged without effect. The denominator, D, of the set is of the penesymmetric type first dealt with in 1868 by Sardi (*Hist.*, iii. pp. 125–126) and is identical with Johnson's of 1877 (*Hist.*, iii. p. 128): the typical numerator is got from D by deleting the first column of the latter and inserting for it a_1,

$a_2, \ldots, a_n$. Among the simple properties of D and N that are discussed we may note the fact that D has $2^n - n$ terms in its final development and N has $(n+1)2^{n-2} - (n-1)$.*

METZLER, W. H. (1901/$_3$)

[On certain aggregates of determinant minors. *Transac. American Math. Soc.*, ii. pp. 395–403.]

The work of this paper in connection with Kronecker's vanishing aggregates has already been referred to, and it only remains for us to note that it is equally effective in regard to Muir's analogue bearing on centrosymmetric determinants. The number and nature of the vanishing aggregates in this case receive critical attention.

NEUBERG, J. (1901/$_6$)

[Question d'examen 1002. *Mathesis*, (3) i. p. 151.]

The subject here is the very simple but interesting penesymmetric

$$\begin{vmatrix} -\dfrac{1}{a} & \dfrac{1}{a+c} & \dfrac{1}{a+b} \\ \dfrac{1}{b+c} & -\dfrac{1}{b} & \dfrac{1}{b+a} \\ \dfrac{1}{c+b} & \dfrac{1}{c+a} & -\dfrac{1}{c} \end{vmatrix}.$$

That it vanishes identically is easily verified by multip columns by $a(b+c)$, $b(c+a)$, $c(a+b)$ respectively.

MUIR, T. (1902^{20}/$_1$)

[The applicability of the Law of Extensible Minors to determinants of special form. *Proceed. Edinburgh Math. Soc.*, xx. pp. 44–49.]

The second of the special determinants referred to here is

* See in Chap. III under Tweedie.

the centrosymmetric, and the theorem established regarding it is: *If each term of a vanishing aggregate of* n-*line minors belonging to a centrosymmetric determinant of the* $(2\mathrm{n})^{th}$ *order be written in the form*

$$\begin{vmatrix} a & b & c, & \dots \\ a, & \beta, & \gamma, & \dots \end{vmatrix}$$

where a, b, c, . . . *are the numbers of the rows and* $\alpha, \beta, \gamma, \dots$ *the numbers of the columns which the minor occupies in the parent determinant, then a generalization (the Extensional) is obtained by changing*

$$\begin{vmatrix} a, & b, & c, & \dots \\ a, & \beta, & \gamma, & \dots \end{vmatrix} \quad \textit{into} \quad \begin{matrix} \vdots & a, & b, & c, & \dots & \vdots \\ \vdots & a, & \beta, & \gamma, & \dots & \vdots \end{matrix}$$

and increasing by n *each number in the latter that is already greater than* n. For example, from knowing the identity (*Hist.*, iv. p. 140)

$$\begin{vmatrix} 1234 \\ 8234 \end{vmatrix} + \begin{vmatrix} 1234 \\ 1734 \end{vmatrix} + \begin{vmatrix} 1234 \\ 1264 \end{vmatrix} + \begin{vmatrix} 1234 \\ 1235 \end{vmatrix}$$

$$= \begin{vmatrix} 8234 \\ 1234 \end{vmatrix} + \begin{vmatrix} 1734 \\ 1234 \end{vmatrix} + \begin{vmatrix} 1264 \\ 1234 \end{vmatrix} + \begin{vmatrix} 1235 \\ 1234 \end{vmatrix}$$

in regard to any 8-line centrosymmetric determinant we can assert the truth of

$$\begin{matrix} \vdots & 1 & 2\;3\;4 & \vdots \\ \vdots & 8+n & 2\;3\;4 & \vdots \end{matrix} + \begin{matrix} \vdots & 1 & 2 & 3\;4 & \vdots \\ \vdots & 1 & 7+n & 3\;4 & \vdots \end{matrix} + \begin{matrix} \vdots & 1\;2 & 3 & 4 & \vdots \\ \vdots & 1\;2 & 6+n & 4 & \vdots \end{matrix} + \begin{matrix} \vdots & 1\;2\;3 & 4 & \vdots \\ \vdots & 1\;2\;3 & 5+n & \vdots \end{matrix}$$

$$= \begin{matrix} \vdots & 8+n & 2\;3\;4 & \vdots \\ \vdots & 1 & 2\;3\;4 & \vdots \end{matrix} + \cdot \quad \cdot \quad \cdot ,$$

it being, however, now carefully noted that the determinant is here extended, not by putting the additional lines after the original lines, but by making them occupy a middle position, with half the original lines before them and the other half after.

NESBITT, A. M. (1904): MUIR, T. (1906³/₉): FITE, W. B. (1906)

(See under these headings in Chapter XI.)

'Anon.' ($1907^1/_4$): KAPADIA, D. D. ($1911/_6$, $1912/_4$)

[Question 566. *Math. Gazette*, iv. p. 56.]

[Questions 297, 373. *Journ. Indian Math. Soc.*, vi. pp. 150–151: iv. p. 193.]

The angles being those of a triangle, the first result here is

$$\begin{vmatrix} \cos(A+2B) & \cos A & \cos A \\ \cos B & \cos(B+2C) & \cos B \\ \cos C & \cos C & \cos(C+2A) \end{vmatrix} = 0.$$

Kapadia's two determinants are also penesymmetric, one of them indeed being essentially identical with Sardi's of 1868. The other is not only new and curious, but has associated with it the odd fact that the evaluational process printed under it at considerable length by the editor was known by him to be incorrect. Its specification is that each of its 1^{st}, 2^{nd}, $(n-1)^{\text{th}}$, and n^{th} rows has a for the element in the diagonal and b for every other element, and that each of the remaining rows has b for the diagonal element and a for every other element. As matters turn out the real problem is to find the cofactor of $(a-b)^{n-1}$ in it: and, since it is centrosymmetric as well as penesymmetric, the intending solver has a choice of two familiar modes of procedure.

MUIR, T. ($1909^8/_1$)

[Waring's expression for a symmetric function in terms of sums of like powers. *Proceed. Edinburgh Math. Soc.*, xxvii. pp. 5–9.]

This historical paper, already referred to, concerns the possible substitution of a penesymmetric determinant for Waring's expression. Hirsch's contribution of 1809 is recalled; the inaccuracies of Bellavitis and of Bruno are explained: and finally a rediscovery is made of Brioschi's result of 1854, the case of the 5^{th} order being used as an illustration (*Hist.*, iii. pp. 373–374: iv. p. 197).

METZLER, W. H. ($1914^1/_3$)

[On centrosymmetric and skew centrosymmetric determinants. *Messenger of Math.*, xliii. pp. 171–176.]

This rediscusses, improves on, and follows up work of Muir's

(Textbook, § 138: *Hist.*, iv. pp. 263–264). In particular it is shown that *In a skew centrosymmetric determinant the sum of the coaxial minors of any odd order is zero*, whence an immediate deduction is that *the Lagrangian equation of a skew centrosymmetric determinant contains only even or only odd powers of the unknown.* The subject of the vanishing aggregates of minors of such a determinant is also considered, and a general theorem given.

MUIR, T. (1914, 1915)

[Question 17726. *Educ. Times*, lxvii. pp. 262, 518: or *Math. from Educ. Times*, (2) xxvii. p. 81.]

[Question 17973. *Educ. Times*, lxviii. pp. 166, 384: or *Math. from Educ. Times*, (2) xxix. pp. 71–72.]

The results here are

$$\begin{vmatrix} a_1 & a_2 & a_3 & x & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & y & \cdot \\ c_1 & c_2 & c_3 & \cdot & \cdot & z \\ x & \cdot & \cdot & a_1 & a_2 & a_3 \\ \cdot & y & \cdot & b_1 & b_2 & b_3 \\ \cdot & \cdot & z & c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 + x & a_2 & a_3 \\ b_1 & b_2 + y & b_3 \\ c_1 & c_2 & c_3 + z \end{vmatrix} \begin{vmatrix} a_1 - x & a_2 & a_3 \\ b_1 & b_2 - y & b_3 \\ c_1 & c_2 & c_3 - z \end{vmatrix},$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & x_4 & x_5 & x_6 \\ b_1 & b_2 & b_3 & y_4 & y_5 & y_6 \\ c_1 & c_2 & c_3 & z_4 & z_5 & z_6 \\ -x_4 & -x_5 & -x_6 & a_1 & a_2 & a_3 \\ -y_4 & -y_5 & -y_6 & b_1 & b_2 & b_3 \\ -z_4 & -z_5 & -z_6 & c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1i+x_4 & a_2i+x_5 & a_3i+x_6 \\ b_1i+y_4 & \cdot\;\cdot\;\cdot & \\ c_1i+z_4 & \cdot\;\cdot\;\cdot & \end{vmatrix} \begin{vmatrix} -a_1i+x_4 & -a_2i+x_5 & -a_3i+x_6 \\ -b_1i+y_4 & & \cdot\;\cdot\;\cdot \\ -c_1i+z_4 & & \cdot\;\cdot\;\cdot \end{vmatrix}.$$

The first 6-line determinant can by transformations of rows

and of columns be expressed as a centrosymmetric: and the second, though only partially centrosymmetric when similarly treated, is seen to be then resolvable by a slightly altered process. The second is expressible as the sum of two squares.*

SIMANDL, V. (1914)

[Vyčisleni zvláštniho determinantu. *Časopis pro pěstováni math. a fys.*, xliv. pp. 43–46.]

The determinant here is of the $(2n)^{\text{th}}$ order, and is a function of three variables, a, b, c, say. The elements of the main diagonal are all equal to a: the parallel diagonal through the $(2, 1)^{\text{th}}$ place is

$$b,\ 2b,\ 3b,\ \ldots,\ (2n-1)b,$$

and through the $(1, 2)^{\text{th}}$ place the same read backwards: the parallel diagonal through the $(3, 1)^{\text{th}}$ place is

$$c,\ c,\ 2c,\ 2c,\ \ldots,\ (n-1)c,\ (n-1)c,$$

and through the $(1, 3)^{\text{th}}$ place the same read backwards: all the other elements are zeros. The property established is linear resolvability. Taking, for example, the 6-line instance

$$\begin{vmatrix} a & 5b & 2c & \cdot & \cdot & \cdot \\ b & a & 4b & 2c & \cdot & \cdot \\ c & 2b & a & 3b & c & \cdot \\ \cdot & c & 3b & a & 2b & c \\ \cdot & \cdot & 2c & 4b & a & b \\ \cdot & \cdot & \cdot & 2c & 5b & a \end{vmatrix}$$

and performing the operations

$$\left.\begin{array}{l} \text{row}_1 - 2\,\text{row}_3 + \text{row}_5 \\ \text{row}_2 - 2\,\text{row}_4 + \text{row}_6 \\ \text{row}_3 - \text{row}_5 \\ \text{row}_4 - \text{row}_6 \end{array}\right\} \text{ followed by } \left\{\begin{array}{l} \text{col}_6 + \text{col}_4 + \text{col}_2 \\ \text{col}_5 + \text{col}_3 + \text{col}_1 \\ \text{col}_4 + 2\,\text{col}_2 \\ \text{col}_3 + 2\,\text{col}_1 \end{array}\right.$$

* An omitted result of 1904 may be inserted although only partly relevant: *The sum of the signed primary minors of a centrosymmetric determinant is equal to a similar determinant of the next lower order and therefore is resolvable into two factors* (*Proceed. R. Soc. Edinburgh*, xxv. p. 373).

we obtain

$$\begin{vmatrix} a-2c & b & \cdot & \cdot & \cdot & \cdot \\ b & a-2c & \cdot & \cdot & \cdot & \cdot \\ c & 2b & a & 3b & \cdot & \cdot \\ \cdot & c & 3b & a & \cdot & \cdot \\ \cdot & \cdot & 2c & 4b & a+2c & 5b \\ \cdot & \cdot & \cdot & 2c & 5b & a+2c \end{vmatrix}$$

which

$$=(a-2c+b)(a-2c-b)(a+3b)(a-3b)(a+2c+5b)(a+2c-5b)$$

or $\quad =(a+5b+2c)(a+3b)(a+b-2c)(a-b-2c)(a-3b)(a-5b+2c).$

In the case of the 8-line determinant the operations are

$$\left.\begin{array}{l} (\mathrm{row}_1,\ \mathrm{row}_3,\ \mathrm{row}_5,\ \mathrm{row}_7 \between 1,\ -3,\ 3,\ -1) \\ (\mathrm{row}_2,\ \mathrm{row}_4,\ \mathrm{row}_6,\ \mathrm{row}_8 \between 1,\ -3,\ 3,\ -1) \\ \qquad (\mathrm{row}_3,\ \mathrm{row}_5,\ \mathrm{row}_7 \between 1,\ -2,\ 1) \\ \qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}$$

$$\text{and}\ \left\{\begin{array}{l} (\mathrm{col}_8,\ \mathrm{col}_6,\ \mathrm{col}_4,\ \mathrm{col}_2 \between 1,\ 1,\ 1,\ 1), \\ (\mathrm{col}_7,\ \mathrm{col}_5,\ \mathrm{col}_3,\ \mathrm{col}_1 \between 1,\ 1,\ 1,\ 1) \\ \qquad (\mathrm{col}_6,\ \mathrm{col}_4,\ \mathrm{col}_2 \between 1,\ 2,\ 1) \\ \qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right.$$

and the determinant equals

$$(a+7b+3c)\,(a+5b+c)\,(a+3b-c)\,(a+b-3c)$$
$$\cdot\,(a-7b+3c)\,(a-5b+c)\,(a-3b-c)\,(a-b-3c).$$

We note for ourselves that although Simandl's determinant is centrosymmetric and therefore can be expressed as a product of two n-line determinants, we are not thus led more expeditiously to the final factorization. For example, the 8-line determinant is equal to

$$\begin{vmatrix} a & 7b & 3c & \cdot \\ b & a & 6b & 3c \\ c & 2b & a & 5b+2c \\ \cdot & c & 3b+2c & a+4b \end{vmatrix} \cdot \begin{vmatrix} a & 7b & 3c & \cdot \\ b & a & 6b & 3c \\ c & 2b & a & 5b-2c \\ \cdot & c & 3b-2c & a-4b \end{vmatrix},$$

but beyond the next stage

$$(a+7b+3c)\begin{vmatrix} a & 7b & 3c & 1 \\ b & a & 6b & 1 \\ c & 2b & a & 1 \\ . & c & 3b+2c & 1 \end{vmatrix} . (a-7b+3c)\begin{vmatrix} a & 7b & 3c & 1 \\ b & a & 6b & 1 \\ c & 2b & a & 1 \\ . & c & 3b-2c & 1 \end{vmatrix}$$

the procedure has little to commend it.

MUIR, T. ($1914^1/_7$): ROSS, C. M. ($1915/_9$)

[Question 17774. *Educ. Times*, lxvii. pp. 354, 386: or *Math. from Educ. Times*, (2) xxvii. pp. 38–39.]

[Question 18069. *Educ. Times*, lxviii. p. 355: or *Math. Quest. and Sol.*, ii. p. 14.]

The six-line centrosymmetric determinant whose first three rows are

$$\begin{matrix} 1 & a & a^2 & \dots & a^5 \\ 1 & b & b^2 & \dots & b^5 \\ 1 & c & c^2 & \dots & c^5 \end{matrix}$$

is here resolved in two different ways into 18 rational factors. The result may be written so as to hold for a $2n$-line determinant, namely:

$$\Pi(1-a^2)(1-ab)^2(a-b)^2.$$

Ross' result is essentially the same as W. W. Johnson's of 1877 (*Hist.*, iii. p. 128) referred to above.

MUIR, T. ($1916^1/_{12}$)

[Question 18339. *Math. Quest. and Sol.*, iii. pp. 54–55.]

The result here established is that if

$$\begin{vmatrix} a-b & c-d \\ a'-b' & c'-d' \end{vmatrix}, \quad \begin{vmatrix} a-c & b-d \\ a'-c' & b'-d' \end{vmatrix}, \quad \begin{vmatrix} a-d & b-c \\ a'-d' & b'-c' \end{vmatrix}$$

be denoted by α, β, γ, then the bisymmetric determinant

$$\begin{vmatrix} \alpha & \beta & \gamma & . \\ \beta & \alpha & . & \gamma \\ \gamma & . & \alpha & \beta \\ . & \gamma & \beta & \alpha \end{vmatrix} = 2^4 \begin{vmatrix} 1 & a & a' \\ 1 & b & b' \\ 1 & c & c' \end{vmatrix} \begin{vmatrix} 1 & a & a' \\ 1 & b & b' \\ 1 & d & d' \end{vmatrix} \begin{vmatrix} 1 & a & a' \\ 1 & c & c' \\ 1 & d & d' \end{vmatrix} \begin{vmatrix} 1 & b & b' \\ 1 & c & c' \\ 1 & d & d' \end{vmatrix}.$$

MUIR, T. ($1919^1/_1$)

[Note on unimodular and other persymmetric determinants. *Transac. R. Soc. S. Africa*, viii. pp. 95–100.]

Incidentally there is here established a companion to Zehfuss' original theorem of 1862 (*Hist.*, iii. p. 124), namely: *Any* n*-line determinant having the array of its last* n — 1 *rows centrosymmetric is expressible as the product of two determinants*: for example, when the order is even

$$\begin{vmatrix} u & v & w & x & y & z \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_3 & c_2 & c_1 \\ b_6 & b_5 & b_4 & b_3 & b_2 & b_1 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ b_6 + b_1 & b_5 + b_2 & b_4 + b_3 \\ a_6 + a_1 & a_5 + a_2 & a_4 + a_3 \end{vmatrix} \cdot \begin{vmatrix} u - z & v - y & w - x \\ b_6 - b_1 & b_5 - b_2 & b_4 - b_3 \\ a_6 - a_1 & a_5 - a_2 & a_4 - a_3 \end{vmatrix};$$

and when the order is odd

$$\begin{vmatrix} u & v & w & x & y \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix} = \begin{vmatrix} b_5 - b_1 & b_4 - b_2 \\ a_5 - a_1 & a_4 - a_2 \end{vmatrix} \cdot \begin{vmatrix} w_3 & v + x & u + y \\ a_3 & a_4 + a_2 & a_5 + a_1 \\ b_3 & b_4 + b_2 & b_5 + b_1 \end{vmatrix}.$$

CHAPTER VI

ALTERNANTS, FROM 1896 TO 1919

In the period in question the interest taken in the study of Alternants shows comparatively little sign of slackening, especially if the years given over to the War be taken into account. Close on seventy (70) writings, in eight different languages and of various degrees of importance, fall to be considered.

One noteworthy feature of the collection is the exceptional number of the papers occupied with the investigation of Symmetric Functions: approximately a third of the whole may be so classified. It is also somewhat striking in connection therewith to note that this body of work has two quite independent sources, the one centring in a small East-Prussian town, the other confined mainly to a busy city in the State of New York. The first efforts of the latter, which were published in 1899, were followed by a more ambitious study in 1901, and the work was continued at intervals until well into 1904. Quite evidently the results set forth in this series of papers were freshly obtained, and were written without any knowledge of the existence of at least one important contribution of about twenty years earlier. At all events no reference is made in them anywhere to the author of the said contribution. In 1906, however, the author himself returned to his old subject, beginning with a paper recalling his memoir of 1881, and following it up during the years 1907, 1908, with four additional contributions. Strange to say, in this new set of five the existence of the American set of five is never even hinted at. Such aloofness and self-sufficiency is of course much to be regretted; it is an inconvenience and a loss to the neutral reader, and a distinct hindrance to the progress of the subject.

It only remains to add that in America since 1904 the research has not been resumed. In Prussia, on the other hand, the aged author in 1917 returned a second time to the work, more convinced than ever of the extreme importance of determinants

as an instrument in the investigation of the properties of Symmetric Functions. In 1918 he added a further short paper, and then finally in 1919 he published a valuable eighty-page memoir summing up his life-work on the subject. He died on 28th December, 1921.

STRÉKALOF, V. DE ($1896/_4$): CARDOSO-LAYNES, G. ($1899^{20}/_{10}$)

[Question 1724. *Nouv. Annales de Math.*, (3) xv. p. 200.]

[Quistione 472. *Periodico di Mat.*, xv. pp. 80, 123.]

Here two alternants are stated to be equal to

$$1!\ 2!\ 3!\ \ldots\ n! \quad \text{or} \quad 1^n \,.\, 2^{n-1}\ \ldots\ (n-1)^2 \,.\, n^1$$

namely,

$$\left| (a-n+r-1)^{s-1} \right|_{n+1} \quad \text{or} \quad \left| (a-n)^0 (a-n+1)^1 \ldots a^n \right|$$

and

$$\left| r^s \right|_n \quad \text{or} \quad \left| 1^1\ 2^2\ 3^3\ \ldots\ n^n \right|,$$

the latter being already familiar.

STUDNIČKA, F. J. ($1900^{16}/_2$)

[O fakultních součinitelích. *Rozpravy České Akad.*, 1900, No. 17, 10 pp.]

In this paper on "Facultätscoefficienten"—simply and clearly written as usual—Studnička expresses any coefficient of the expansion of

$$(x+a_1)(x+a_2)\ldots(x+a_n)$$

as the quotient of two alternants (*Hist.*, ii. pp. 175–176), and makes evaluations like

$$\left| 2^0\ 5^1\ 8^3\ 11^5 \right| = \zeta^{\frac{1}{2}}(2,\ 5,\ 8,\ 11) \cdot \begin{vmatrix} 26 & 1 \\ 806 & 231 \end{vmatrix}$$

as taught by Naegelsbach (*Hist.*, iii. pp. 145–146).

MUIR, T. (1900$^{19}/_{3}$)

[On Jacobi's expansion for the difference-product when the number of elements is even. *Proceed. R. Soc. Edinburgh*, xxiii. pp. 133–141.]

The nature of the expansion in question is seen from the cases of the 4th and 6th orders

$$|a^0b^1c^2d^3| = \Sigma(b-a)(d-c)(a^2b^2+c^2d^2),$$

$$|a^0b^1c^2d^3e^4f^5| = \Sigma(b-a)(d-c)(f-e)(a^4b^4c^2d^2+\ldots),$$

the remaining terms being got from the typical term by the use of cyclical substitutions (*Hist.*, i. pp. 329–330). Besides supplying a needed proof Muir discovers in a Pfaffian a simple source of one part of each term and in a permanent an equally simple source of the other part, and thence formulates and proves the theorem, *The difference-product of* 2n *elements may be expressed as an aggregate of* (2n — 1) (2n — 3) . . . 3 . 1 *terms, which are obtainable by taking the ordinary expansion of the Pfaffian of the* n(2n — 1) *differences arranged in the ordinary triangular fashion, and then annexing to each term of this expansion an alternant-like permanent whose diagonal elements are the* 0*th*, 2*nd*, 4*th*, . . . *powers respectively of the products of the two original elements occurring in each of the linear factors of the term.* For example

$$|a^0b^1c^2d^3e^4f^5| = \Sigma(b-a)(d-c)(f-e)\begin{vmatrix} 1 & ab & a^2b^2 \\ 1 & cd & c^2d^2 \\ 1 & ef & e^2f^2 \end{vmatrix}^{+}_{+}$$

where $(b-a)(d-c)(f-e)$ is taken with its appropriate sign from

$$\begin{vmatrix} b-a & c-a & d-a & e-a & f-a \\ & c-b & d-b & e-b & f-b \\ & & d-c & e-c & f-c \\ & & & e-d & f-d \\ & & & & f-e \end{vmatrix}.$$

The remainder of the paper concerns the "rule-of-signs" in Pfaffians and general determinants.

MUIR, T. ($1900^{19}/_3$)

[The theory of alternants in the historical order of development up to 1841. *Proceed. R. Soc. Edinburgh*, xxiii. pp. 93–132.]

This, the first of our papers on the history of alternants, although it deals with only six writings, extends to 40 pages; but it must be remembered that among the writers are Cauchy, Schweins, Jacobi, all of whom were large contributors to the theory.

JUNG, V. ($1900/_7$)

[Poznámka o jistém determinatu mocninném. *Časopis pro pěstov. math. a fys.*, xxix. pp. 41–42.]

Here the curious theorem, that the determinant

$$\begin{vmatrix} 1^1 & 1^2 & \dots & 1^n & 1^{n+1} \\ 2^1 & 2^2 & \dots & 2^n & 2^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n^1 & n^2 & \dots & n^n & n^{n+1} \\ \frac{n^1}{2} & \frac{n^2}{3} & \dots & \frac{n^n}{n+1} & \frac{n^{n+1}}{n+2} \end{vmatrix}$$

vanishes when n is even, is verified in an unsuggestive way for the first three cases.

ZEMPLÉN, G. ($1900^1/_{12}$)

[Adalék az interpoláczió és a pareziális törtek elméletéhez. *Math. és Phys. Lapok*, ix. pp. 386–404: or, in French, *Archiv d. Math. u. Phys.*, (3) viii. pp. 214–226.]

As an auxiliary to his main subject the author here restates Beke's theorem of 1898 regarding alternants with differentiated rows (*Hist.*, iv. p. 197), apparently being unaware of the work of any of Beke's predecessors, Schendel, Weihrauch, Méray, ... (*Hist.*, iv. p. 201).

Magyar interest in such determinants seems to date from the year of the starting of the *Math. és Phys. Lapok.* In the first volume (1891–1892) Rados set for evaluation a determinant like

Besso's of 1882 (*Hist.*, iv. p. 155): and in the second volume (pp. 161–172) appeared five solutions,* one of the contributors being Beke, who, as we have seen, continued afterwards the study of the subject.

MUIR, T. ($1900^3/_{12}$)

[Some identities connected with alternants and with elliptic functions. *Transac. R. Soc. Edinburgh*, xl. pp. 187–201.]

These interesting identities are Cayley's of the year 1849,† and were given by him without proof. They are

$$\begin{vmatrix} 1 & a & A \\ 1 & b & B \\ 1 & c & C \end{vmatrix} (B+C)(C+A)(A+B)$$

$$= \begin{vmatrix} 1 & a & A^2 \\ 1 & b & B^2 \\ 1 & c & C^2 \end{vmatrix} (\Sigma A^2 + \Sigma BC) - \begin{vmatrix} 1 & a & A^4 \\ 1 & b & B^4 \\ 1 & c & C^4 \end{vmatrix},$$

$$\begin{vmatrix} 1 & a & aA \\ 1 & b & bB \\ 1 & c & cC \end{vmatrix} (B+C)(C+A)(A+B)$$

$$= \begin{vmatrix} 1 & a & aA^2 \\ 1 & b & bB^2 \\ 1 & c & cC^2 \end{vmatrix} (\Sigma A^2 + \Sigma BC) - \begin{vmatrix} 1 & a & aA^4 \\ 1 & b & bB^4 \\ 1 & c & cC^4 \end{vmatrix},$$

$$\begin{vmatrix} 1 & a & a^2 & aA \\ 1 & b & b^2 & bB \\ 1 & c & c^2 & cC \\ 1 & d & d^2 & dD \end{vmatrix} L = \begin{vmatrix} 1 & a & a^2 & aA^2 \\ 1 & b & b^2 & bB^2 \\ 1 & c & c^2 & cC^2 \\ 1 & d & d^2 & dD^2 \end{vmatrix} M$$

$$- \begin{vmatrix} 1 & a & a^2 & aA^4 \\ 1 & b & b^2 & bB^4 \\ 1 & c & c^2 & cC^4 \\ 1 & d & d^2 & dD^4 \end{vmatrix} N + \begin{vmatrix} 1 & a & a^2 & aA^6 \\ 1 & b & b^2 & bB^6 \\ 1 & c & c^2 & cC^6 \\ 1 & d & d^2 & dD^6 \end{vmatrix} P,$$

* Zemplén states that the value of the determinant is also given on p. 168 of Markoff's *Differenzenrechnung* of 1896, the Russian original of which appeared in 1889.

† Cayley, A., " Note sur l'addition des fonctions elliptiques ", *Crelle's Journ.*, xli. pp. 57–65; or *Collected Math. Papers*, i. pp. 540–549.

$$\begin{vmatrix} 1 & a & A & aA \\ 1 & b & B & bB \\ 1 & c & C & cC \\ 1 & d & D & dD \end{vmatrix} L = \begin{vmatrix} 1 & a & A^2 & aA^2 \\ 1 & b & B^2 & bB^2 \\ 1 & c & C^2 & cC^2 \\ 1 & d & D^2 & dD^2 \end{vmatrix} Q - \begin{vmatrix} 1 & a & A^4 & aA^4 \\ 1 & b & B^4 & bB^4 \\ 1 & c & C^4 & cC^4 \\ 1 & d & D^4 & dD^4 \end{vmatrix},$$

where

$$L = (A+B)(A+C)(A+D)(B+C)(B+D)(C+D),$$
$$M = \Sigma A^3B^2 + \Sigma A^3BC + 2\Sigma A^2B^2C + 3\Sigma A^2BCD,$$
$$N = \Sigma A^3 + \Sigma A^2B + \Sigma ABC, \qquad P = \Sigma A,$$
$$Q = \Sigma A^2B + \Sigma A^2BC + 2ABCD.$$

Muir gives four different modes of proof, three of which are dependent on an important transformation got by substituting for each specified symmetric function a quotient of two alternants. If for shortness' sake we represent each determinant by its first row only, the identities then take the form

$$|1 \quad a \quad A|\,|1 \quad A^2 \quad A^4| - |1 \quad a \quad A^2|\,|1 \quad A \quad A^4| + |1 \quad a \quad A^4|\,|1 \quad A \quad A^2| = 0,$$

$$|1 \quad a \quad aA|\,|1 \quad A^2 \quad A^4| - |1 \quad a \quad aA^2|\,|1 \quad A \quad A^4| + |1 \quad a \quad aA^4|\,|1 \quad A \quad A^2| = 0,$$

$$|1\; a\; a^2\; aA|\,|1\; A^2\; A^4\; A^6| - |1\; a\; a^2\; aA^2|\,|1\; A\; A^4\; A^6|$$
$$+ |1\; a\; a^2\; aA^4|\,|1\; A\; A^2\; A^6| - |1\; a\; a^2\; aA^6|\,|1\; A\; A^2\; A^4| = 0,$$

$$|1\; a\; A\; aA|\,|1\; A^2\; A^4\; A^6| - |1\; a\; A^2\; aA^2|\,|1\; A\; A^4\; A^5|$$
$$+ |1\; a\; A^4\; aA^4|\,|1\; A\; A^2\; A^3| = 0.$$

The first two are shown to be included in the manifest identity

$$\begin{vmatrix} h & m & aA_1 \\ k & n & bB_1 \\ l & r & cC_1 \end{vmatrix} |1 \quad A_2 \quad A_4| - \begin{vmatrix} h & m & aA_2 \\ k & n & bB_2 \\ l & r & cC_2 \end{vmatrix} |1 \quad A_1 \quad A_4|$$

$$+ \begin{vmatrix} h & m & aA_4 \\ k & n & bB_4 \\ l & r & cC_4 \end{vmatrix} |1 \quad A_1 \quad A_2| - \begin{vmatrix} h & m & a \\ k & n & b \\ l & r & c \end{vmatrix} |A_1 \quad A_2 \quad A_4|$$

$$
= \begin{vmatrix} h & m & aA_1 & aA_2 & aA_4 & a \\ k & n & bB_1 & bB_2 & bB_4 & b \\ l & r & cC_1 & cC_2 & cC_4 & c \\ \cdot & \cdot & A_1 & A_2 & A_4 & 1 \\ \cdot & \cdot & B_1 & B_2 & B_4 & 1 \\ \cdot & \cdot & C_1 & C_2 & C_4 & 1 \end{vmatrix} = 0;
$$

and the third in the similar identity got from

$$
\begin{vmatrix} m_1 & m_2 & m_3 & aA_1 & aA_2 & aA_4 & aA_6 & aA_0 \\ n_1 & n_2 & n_3 & bB_1 & bB_2 & bB_4 & bB_6 & bB_0 \\ r_1 & r_2 & r_3 & cC_1 & cC_2 & cC_4 & cC_6 & cC_0 \\ s_1 & s_2 & s_3 & dD_1 & dD_2 & dD_4 & dD_6 & dD_0 \\ \cdot & \cdot & \cdot & A_1 & A_2 & A_4 & A_6 & A_0 \\ \cdot & \cdot & \cdot & B_1 & B_2 & B_4 & B_6 & B_0 \\ \cdot & \cdot & \cdot & C_1 & C_2 & C_4 & C_6 & C_0 \\ \cdot & \cdot & \cdot & D_1 & D_2 & D_4 & D_6 & D_0 \end{vmatrix} = 0.
$$

It is otherwise, however, with the last of the four, and to this other methods are applied, one of them consisting in performing the indicated multiplications with the result

$$
\begin{vmatrix} 4 & \Sigma a & \Sigma A & \Sigma aA \\ \Sigma A^2 & \Sigma aA^2 & \Sigma A^3 & \Sigma aA^3 \\ \Sigma A^4 & \Sigma aA^4 & \Sigma A^5 & \Sigma aA^5 \\ \Sigma A^6 & \Sigma aA^6 & \Sigma A^7 & \Sigma aA^7 \end{vmatrix} + \begin{vmatrix} 4 & \Sigma a & \Sigma A^4 & \Sigma aA^4 \\ \Sigma A & \Sigma aA & \Sigma A^5 & \Sigma aA^5 \\ \Sigma A^2 & \Sigma aA^2 & \Sigma A^6 & \Sigma aA^6 \\ \Sigma A^3 & \Sigma aA^3 & \Sigma A^7 & \Sigma aA^7 \end{vmatrix}
$$

$$
+ \begin{vmatrix} 4 & \Sigma a & \Sigma A^2 & \Sigma aA^2 \\ \Sigma A & \Sigma aA & \Sigma A^3 & \Sigma aA^3 \\ \Sigma A^4 & \Sigma aA^4 & \Sigma A^6 & \Sigma aA^6 \\ \Sigma A^5 & \Sigma aA^5 & \Sigma A^7 & \Sigma aA^7 \end{vmatrix} = 0
$$

and then showing that this is a special case of the identity

$$
\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & c_1 & c_2 \\ a_3 & a_4 & c_3 & c_4 \\ b_1 & b_2 & d_1 & d_2 \\ b_3 & b_4 & d_3 & d_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{vmatrix} = 0.
$$

The paper excites increased interest as to the corresponding identities of higher order.

GIUDICE, F. ($1901^{10}/_5$)

[Sul resto della divisione algebrica. *Periodico di Mat.*, Anno xvii. pp. 88–90.]

Here the divisor of $\Phi(x)$ being in the exceptional form $(x-a)(x-b)(x-c)(x-d)$, the guiding equality is

$$\Phi(x) = (x-a)(x-b)(x-c)(x-d)\cdot Q(x) + R(x),$$

and there is thence found

$$R(x) = -\begin{vmatrix} . & x^3 & x^2 & x & 1 \\ \Phi(a) & a^3 & a^2 & a & 1 \\ \Phi(b) & b^3 & b^2 & b & 1 \\ \Phi(c) & c^3 & c^2 & c & 1 \\ \Phi(d) & d^3 & d^2 & d & 1 \end{vmatrix} \div \left| a^3b^2c^1d^0 \right|,$$

which, since $\Phi(a)$, $\Phi(b)$, ... are the same as $R(a)$, $R(b)$, ... respectively, is seen to be also the solution of the interpolation-problem of finding a function $R(x)$ which shall equal $\Phi(a)$, $\Phi(b)$, ... when x equals a, b, ...

The other result given dates at least as far back as Bellavitis (1857)* (*Hist.*, ii. p. 181).

VOGT, H. ($1901/_8$)

[Sur l'apolarité des formes binaires. *Nouv. Annales de Math.*, (4) i. pp. 337–365.]

Incidentally Vogt here obtains a result which, unlike Zemplén's just reported on, deserves careful comparison and contrast with those of the type dealt with by Schendel and others. It is

$$\begin{vmatrix} m_5x^5 & (m-1)_4y^4 & m_5y^5 & (m-2)_3z^3 & (m-1)_4z^4 & m_5z^5 \\ m_4x^4 & (m-1)_3y^3 & m_4y^4 & (m-2)_2z^2 & (m-1)_3z^3 & m_4z^4 \\ m_3x^3 & (m-1)_2y^2 & m_3y^3 & (m-2)_1z & (m-1)_2z^2 & m_3z^3 \\ m_2x^2 & (m-1)_1y & m_2y^2 & 1 & (m-1)_1z & m_2z^2 \\ m_1x & 1 & m_1y & . & 1 & m_1z \\ 1 & . & 1 & . & . & 1 \end{vmatrix}$$

$$= m_5\cdot(m-1)_3\cdot m_3\cdot(x-y)^2(x-z)^3(y-z)^2,$$

* The reference made to Retali (*Le Mat. pure ed appl.*, i. pp. 14–16) is unimportant.

where it is seen that the right-hand member contains a factor in m which is absent from the corresponding determinant of the type referred to (*Hist.*, iv. p. 179). A point to be at once noted is that, though thus outwardly seeming to be more general than Schendel's, it is not so in the sense that specialization of the extra variable will give the companion result. On the other hand there is a latent point which amounts to much in establishing resemblance between the two, namely, that differentiation plays the same part in the construction and factorization of both. For example, in the simpler one differentiation of the 6^{th} row gives the 5^{th} row, and in the other differentiation of the 6^{th} row gives m times the 5^{th}—a distinction of no moment when our only aim is to be able to assert that, if the 5^{th} column and the column resulting from differentiation of the 6^{th} exist in the same determinant that determinant must vanish.

ROE, E. D. ($1901^{25}/_{10}$)

[Note on symmetric functions. *American Journ. of Math.*, xxv. pp. 97–106.]

The subject of this note is the already much-discussed theorem regarding the multiplication of $|a^0b^1c^2 \ldots|_n$ by a symmetric function of the variables, and Kostka's corollary therefrom of 1875 (*Hist.*, iii. pp. 154–156). What the writer seeks to expedite is the finding of the coefficient of any separate term of the product. To this end he gives an interesting theorem, generalized by him in his next paper, but, unfortunately, not recognized as one of the properties of Kostka's tables of 1881. For example, it being known from any source that in the product of $|\alpha^0\beta^1\gamma^2\delta^3\epsilon^4|$ by $\Sigma\alpha^3\beta^2$ there occurs the alternant $|\alpha^0\beta^2\gamma^3\delta^4\epsilon^6|$, the theorem affirms that the coefficient of it is the coefficient of $\Sigma\alpha\beta\gamma \cdot \Sigma\alpha\beta$, or say $c_3c_2c_0c_0c_0$, in

$$\begin{vmatrix} c_0 & c_2 & c_3 & c_4 & c_6 \\ \cdot & c_1 & c_2 & c_3 & c_5 \\ \cdot & c_0 & c_1 & c_2 & c_4 \\ \cdot & \cdot & c_0 & c_1 & c_3 \\ \cdot & \cdot & \cdot & c_0 & c_2 \end{vmatrix},$$

the finding of which is readily accomplished by judiciously re-

ducing the order of the determinant.* The calculator also obtains from him a hint that, if *all* the terms of the product be wanted, it may be easier to find the determinants of the corollary first and thence those of the product.

METZLER, W. H. ($1902^2/_6$)

[Some identities connected with alternants and with elliptic functions. *Proceed. R. Soc. Edinburgh*, xxiv. pp. 240–243.]

These are again Cayley's identities of 1849 (see above under 1900); and the new step taken in regard to them is to lay at rest one of Muir's questions as to possible generalization, proof being adduced that there can be no identity of the form

$$
\begin{aligned}
&|\,1\;a\;a^2\;A\;aA\,| \cdot |\,A^0B^2C^4D^6E^8\,| \\
= &|\,1\;a\;a^2\;A^2\;aA^2| \, |A^0B^\alpha C^\beta D^\gamma E^\delta| \pm |1\;a\;a^2\;A^4\;aA^4| \, |A^0B^\epsilon C^\zeta D^\eta E^\theta| \\
&\pm |\,1\;a\;a^2\;A^6\;aA^6| \, |A^0B^1C^2D^3E^4|.
\end{aligned}
$$

TAYLOR, W. E. ($1902^{15}/_{12}$)

[On the product of an alternant by a symmetric function. *American Math. Monthly*, x. pp. 119–130.]

The practical outcome of this paper is an instalment of multiplication tables for use in applying the method suggested and illustrated by Muir in 1899 for finding the product in question (*Hist.*, iv. p. 198). In the first table the constant multiplicand is $|\,\alpha_1{}^0\alpha_2{}^1 \ldots \alpha_n{}^{n-1}\,|$ and the multipliers (44 in number of the form $c_1{}^uc_2{}^vc_3{}^w \ldots$) extend from $\Sigma\alpha_1$ to $\Sigma\alpha_1\alpha_2 \ldots \alpha_6\alpha_7$. In the second the multipliers are the first eleven of the form

$$\Sigma\alpha_1\alpha_2 \ldots \alpha_{n-r}(\Sigma\alpha_1)^r.$$

Some interest also attaches to the account given of the procedure followed in the formation of the tables. The calculations made form a check on part of Kostka's.

* Also, we may add, by at once putting c_1, c_4, c_5, c_6 equal to 0.

MUIR, T. ($1903^{28}/_{2}$)

[Historical note on determinants. *Nature*, lxvii. p. 512.]

This was called forth by a curiously misleading statement in Roe's paper of 1901 in regard to the authorship of two theorems. In it the injustice done to Schweins, in particular, is very pointedly dwelt on.

ROE, E. D. ($1903^{16}/_{11}$)

[On the coefficients in the product of an alternant and a symmetric function. *Transac. American Math. Soc.*, v. pp. 193–213.]

The contents of this highly symbolized paper are not fully indicated in the title, there being in it considerable matter, other than determinantal, concerning symmetric functions. What first calls for attention is the generalized theorem above referred to, namely, *If* $| a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_m^{\lambda_m} | \cdot \Sigma a_1^{p_1} a_2^{p_2} \ldots a_m^{p_m}$ *be expressed as an aggregate of alternants, the coefficient of any one of these alternants* $| a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m} |$ *is the same as the coefficient of* $c_{p_1} c_{p_2} \ldots c_{p_m}$ *in the final expansion of*

$$| c_{k_1-\lambda_1} c_{k_2-\lambda_2} \quad \ldots \quad c_{k_m-\lambda_m} |,$$

the c*'s here being the simple symmetric functions* Σa_1, $\Sigma a_1 a_2$, . . . The use of this in practice is illustrated, tables of the coefficients are constructed, and the applicability of the tables in finding the quotient of an alternant by the difference-product is pointed out. Of less special interest are the sections on aleph functions and on the utility of the tables for the additional purpose of the common so-called "symmetric-function" table—that is to say, the table giving $\Sigma \alpha^p \beta^q \gamma^r \ldots$ in terms of $\Sigma \alpha$, $\Sigma \alpha\beta$, . . . It must be noted, however, that the tables thus dwelt upon are included in the first seven of Kostka's of 1881 (*Hist.*, iv. Chap. VI), each table being obtainable from Kostka's corresponding table by simply leaving out the half of the latter on the lower side of the diagonal.

KÖNIG, J. (1903)

[Die elementaren symmetrischen Formen. *Einleitung . . . Algebraischen Gröszen*, pp. 46–53.]

In dealing with symmetric functions König in his own way introduces and establishes (pp. 47–53) the known determinantal expression for the difference-product, and follows it up with a wider theorem closely resembling Mertens' second theorem of 1889 (*Hist.*, iv. p. 177). In the latter the determinant dealt with is of the order $(m)_r \cdot r!$, where m is the total number of variables $x_1, x_2, \ldots$, and r is the number of them occurring in any row. It is shown to be equal to a special power of the difference-product $|\, x_1^0 x_2^1 \ldots x_m^{m-1} \,|$. For example, if m is 4 and r is 3 the determinant is of the 24th order; the variables that occur in each of the first six rows are x_1, x_2, x_3, in each of the next six x_1, x_2, x_4, and so on; the elements of the row that has the variables u, v, w are the terms of the expansion of

$$(u^3 + u^2 + u + 1)(v^2 + v + 1)(w + 1):$$

and the value of the determinant is $|\, x_1^0 x_2^1 x_3^2 x_4^3 \,|^{12}$. For the power (12 in this case) of the difference-product no quite satisfactory expression is given: if it be denoted by $N_{m,r}$ we are told that $N_{m,1} = 1$, $N_{m,2} = 2m - 3$, $N_{m,3} = 3(m-2)^2$, and that when $r > 2$

$$N_{m,r} = \tfrac{1}{2} r(2m - r - 1) \cdot (m-2)(m-3) \ldots (m - r + 1).$$

GIAMBELLI, G. Z. (1903)

[Alcune proprietà delle funzioni simmetriche caratteristiche. *Atti . . . Accad. delle Sci.* (Torino), xxxviii. pp. 551–572.]

The main and certainly the fundamental subject of this important paper is those symmetric functions that are quotients of alternants by the difference-product of the variables, and which, as the difference-product itself is an alternant, we found in 1879 convenient and appropriate to call 'bialternant symmetric functions' (*Hist.*, iii. pp. 169–170). Although quite as much as a fourth of the whole paper is preparatory and historical, the value of this as an introduction would have been much enhanced if the author's acquaintance with previous related work

had been full and accurate. As it is, well-known theorems like Jacobi's of 1841 (*Hist.*, i. pp. 341–342) are misattributed, and the work of writers like Kostka, who made a specialty of such symmetric functions (*Hist.*, iv. pp. 145–146, . . .), is not alluded to at all.

The first fresh matter (§ 3) concerns what are called dual groups of integers, an acquaintance with these being necessary for the statement and establishment of the ‘ principle of duality ’, the goal which the writer keeps steadily in view. The section opens with a formal definition and explanation of an extension of a concept which we have already met with in Naegelsbach's theorem of 1871 (*Hist.*, iii. pp. 145–146). In the example there used for illustration the pair of related sets of integers belonging to the sequence 0, 1, 2, 3, 4, 5, 6 are

$$(0, 2, 5, 6) \quad \text{and} \quad (5, 3, 2)$$

the one, (5, 3, 2) say, being got from the other by taking the complementary set of the latter—that is to say, the set (1, 3, 4) —and then substituting for each of the integers 1, 3, 4 the excess of 6 over the said integer. As the excesses 5, 3, 2 are in a sense the complements of 1, 3, 4, the procedure consists in twice taking complements, so that it would be suggestive and appropriate to call (0, 2, 5, 6) and (5, 3, 2) bicomplementary sets. A brief substitute for Giambelli's definition would thus be: *Two sets of integers are said to be bicomplementary when the integers of one set and the excesses of* $m + 1$ *over those of the other form a permutation of the integers from* 1 *to* m *inclusive.* A set of integers may, of course, be its own bicomplementary, for example, the set (2, 5, 6, 8) belonging to the sequence, 1, 2, . . . , 8.

As for the ‘ principle of duality ’ the only thing we have had before now which at all closely resembles it is the Law of Complementaries. According to the latter, it will be remembered, if we have obtained in any way an identical relation between minors of a general determinant, another such relation can at once be got by simply substituting for each minor of the former the complementary minor (*Hist.*, iv. pp. 7–8). Now the points in which Giambelli's law differs from this are: (1) that the entities connected by the presupposed identical relation have now to be bi-alternant symmetric functions of one and the same set of variables, and (2) that for each such function's set of indices there

has to be substituted the bicomplementary set. The close resemblance in form between the two laws does not, however, extend to the methods of proof. In the case of the new law there is a basic theorem whose limitations give it a peculiar interest and induce us to present an illustrative case, namely, the case in which the two sets of variables involved are both 4 in number. In effect it then is: *If the four relations*

$$\left.\begin{aligned} a+b+c+d &= x+y+z+w, \\ \Sigma ab &= (x, y, z, w)^2, \\ \Sigma abc &= (x, y, z, w)^3, \\ abcd &= (x, y, z, w)^4 \end{aligned}\right\}$$

hold, then every bi-alternant symmetric function of a, b, c, d *that is not of a higher degree than the* 4^{th} *is equal to the bicomplementary function of* x, y, z, w. In regard to this our first remark is that the given relations are themselves instances of the equalities to be established, being in determinant notation

$$\begin{aligned} |a^0b^1c^2d^4| \div \zeta_1^{\frac{1}{2}} &= |x^0y^1z^2w^4| \div \zeta_2^{\frac{1}{2}}, \\ |a^0b^1c^3d^4| \div \zeta_1^{\frac{1}{2}} &= |x^0y^1z^2w^5| \div \zeta_2^{\frac{1}{2}}, \\ |a^0b^2c^3d^4| \div \zeta_1^{\frac{1}{2}} &= |x^0y^1z^2w^6| \div \zeta_2^{\frac{1}{2}}, \\ |a^1b^2c^3d^4| \div \zeta_1^{\frac{1}{2}} &= |x^0y^1z^2w^7| \div \zeta_2^{\frac{1}{2}}, \end{aligned}$$

where $\zeta_1^{\frac{1}{2}}$, $\zeta_2^{\frac{1}{2}}$ stand for $|a^0b^1c^2d^3|$, $|x^0y^1z^2w^3|$ respectively, and where we see that each set of indices on the left is the bicomplementary of the corresponding set on the right. It would thus seem that here by enforcing compliance with a law in four instances we ensure compliance in six other instances.* Individually

* If we write our given equations still more curtly in the form

$$\left.\begin{aligned} A(0\,1\,2\,4) &= X(0\,1\,2\,4), \\ A(0\,1\,3\,4) &= X(0\,1\,2\,5), \\ A(0\,2\,3\,4) &= X(0\,1\,2\,6), \\ A(1\,2\,3\,4) &= X(0\,1\,2\,7), \end{aligned}\right\}$$

then the six additional instances are

$$\left.\begin{aligned} A(0\,1\,2\,5) &= X(0\,1\,3\,4), & & \\ A(0\,1\,2\,6) &= X(0\,2\,3\,4), & A(0\,1\,3\,5) &= X(0\,1\,3\,5), \\ A(0\,1\,2\,7) &= X(1\,2\,3\,4), & A(0\,1\,3\,6) &= X(0\,2\,3\,5), \\ & & A(0\,2\,3\,5) &= X(0\,1\,3\,6), \end{aligned}\right\}$$

or at greater length

$$\left.\begin{aligned} &(a, b, c, d)^2 = \Sigma xy, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &\qquad \Sigma a^2bc + 3abcd = \Sigma x^3y + \Sigma x^2y^2 + 2\Sigma x^2yz + 3xyzw \end{aligned}\right\}.$$

these results are readily verifiable: for example, the operation which we may symbolize by (3) — 2(2)(1) + $(1)^3$ gives us

$$\Sigma a^3 + \Sigma a^2 b + \Sigma abc = \Sigma xyz,$$

the counterpart of (3). It must, however, be pointedly noted that with equal convincingness $(1)^2$ — 2(2) gives us

$$a^2 + b^2 + c^2 + d^2 = -x^2 - y^2 - z^2 - w^2$$

and thus suggests the desirability of having an alternative proof for Giambelli's law.

The next section (§ 5), which is devoted to applications of the law, furnishes only two results, one derivable more naturally from Trudi's of 1862 (*Hist.*, iii. p. 214) by interchanging $\aleph$'s and c's, and the other being Naegelsbach's of 1871 already referred to. Neither of them, we may add, is got unaidedly by the direct and simple substitution specified in the statement of the law.

The rest of the paper (§§ 6–8) deals with matters which less directly concern us, ending with a contribution to Enumerative Geometry. It is desirable merely to point out in connection with it that the main theorem dealt with (§ 7) and attributed to H. Schubert is what Bazin's theorem of 1851 becomes when general determinants are replaced by alternants.

ROE, E. D. (1904/$_5$)

[On the coefficients in the quotient of two alternants. *Transac. American Math. Soc.*, vi. pp. 63–74.]

The second of the alternants in question is the difference-product, so that the subject is one already familiarized to us. Aids towards the speedier calculation of the coefficients—recurrence-formulæ and other minor relations—are brought forward: the results of the calculations are tabulated: and additional purposes to which they lend themselves are pointed out. The new tables, like those of the former paper, are in shape right-angled triangles; and as each triangle of the former set is reprinted and united with the corresponding triangle of the new set to form a square, the result now is in effect an exact reproduction of Kostka's first seven tables.

NANSON, E. J. ($1905^1/_{10}$)

[Question 15860. *Educ. Times*, lviii. p. 449; lix. p. 349.]

This is a simple * case of finding the quotient of an alternant by the difference-product of the variables. Note may be taken that the dividend

$$\begin{vmatrix} 1 & (b+c)^2 & b^2c^2 \\ 1 & (c+a)^2 & c^2a^2 \\ 1 & (a+b)^2 & a^2b^2 \end{vmatrix} = 2\zeta^{\frac{1}{2}}(bc, ca, ab) - \zeta^{\frac{1}{2}}(a^2, b^2, c^2).$$

NANSON, E. J. ($1905^1/_{11}$): MUIR, T. ($1906^1/_1$)

[Questions 15880, 15913. *Educ. Times*, lviii. p. 494; lix. pp. 41, 148; or *Math. from Educ. Times*, (2) x. pp. 53–54.]

The alternants

$$\begin{vmatrix} y+z-2x & x^2-yz & 2xyz-x^2(y+z) \\ z+x-2y & y^2-zx & 2xyz-y^2(z+x) \\ x+y-2z & z^2-xy & 2xyz-z^2(x+y) \end{vmatrix},$$

$$\begin{vmatrix} y+z-2x & -2x^2-yz & 2xyz+x^2 \\ z+x-2y & -2y^2-zx & 2xyz+y^2 \\ x+y-2z & -2z^2-xy & 2xyz+z^2 \end{vmatrix}$$

are shown as equal to

$$0, \quad (z-y)(z-x)(y-x).(\Sigma x^3 - 3\Sigma x^2y + 15xyz)$$

by multiplying $|x^0y^1z^2|$ by

$$\begin{vmatrix} \Sigma x & -3 & . \\ -\Sigma xy & \Sigma x & . \\ 3xyz & -\Sigma xy & . \end{vmatrix}, \quad \begin{vmatrix} \Sigma x & -3 & . \\ -\Sigma xy & \Sigma x & -3 \\ 3xyz & -\Sigma xy & \Sigma x \end{vmatrix},$$

respectively.

* Still simpler is a question put in the *Math. Gaz.*, ii. (1903) p. 363, in regard to an alternant equal to $|a^0b^1c^2| . |a^0b^1c^3|$. The equality given in the *Educ. Times*, lxvi. (1913) p. 218, dates back to 1876 at least; and the set of equations dealt with in the *American Math. Monthly*, viii. (1901–1902), p. 266; ix. pp. 136–137, 162, is that known as Lagrange's (*Hist.*, ii. p. 155).

KOSTKA, C. (1906)

[Zur Bildung der symmetrischen Funktionen. *Archiv d. Math. u. Phys.*, (3) x. pp. 50–55.]

This being written by Kostka merely to compare his methods with those of Saalschütz as expounded shortly before in a 30-page paper,* naturally contains nothing fresh on our subject. The superiority which he claims is attributed to the use of determinants.

MUIR, T. ($1906^{30}/_5$)

[The expression of certain symmetric functions as an aggregate of fractions. *Transac. S. African Philos. Soc.*, xvi. pp. 313–315.]

The writer's object is to show that the type of equality established by E. Prouhet (*Nouv. Ann. de Math.*, xv. pp. 86–91) holds for symmetric functions other than Σa, Σab, Σabc, . . . We have only to take an alternant of the form

$$| a_1{}^0 a_2{}^1 \ldots a_m^{m-1}\, a_{m+1}^{s}\, a_{m+2}^{s+1} \ldots |_n, \quad \text{say } | a^0 b^1 c^4 d^5 e^6 |,$$

expand it in terms of the 3-line minors of the last three columns, and then divide by $| a^0 b^1 c^2 d^3 e^4 |$, the result of this being

$$\Sigma a^2bcde - \Sigma a^2b^2c^2 = \Sigma \frac{a^4b^4c^4}{(a-d)(a-e)\,.\,(b-d)(b-e)\,.\,(c-d)(c-e)}.$$

When $s < m$, equalities like

$$1 = \Sigma \frac{a^2b^2c^2}{(a-d)(a-e)\,.\,(b-d)(b-e)\,.\,(c-d)(c-e)},$$

$$0 = \Sigma \frac{abc}{(a-d)(a-e)\,.\,(b-d)(b-e)\,.\,(c-d)(c-e)}.$$

are obtained. The extension of the procedure to even-ordered alternants like

$$| a^0\, b^1\, c^r\, d^{r+1}\, e^s f^{s+1} |$$

leads to results including Jacobi's (see above, p. 178).†

* *Archiv* (3), ix. pp. 113–143.

† An omitted result of 1904 may here be inserted, although out of its proper place: The sum of the signed primary minors of the alternant $| a^0 b^\alpha c^\beta | \ldots$ is equal to the alternant itself (*Proceed. R. Soc. Edinburgh*, xxv. p. 372).

MUIR, T. ($1906^2/_7$)

[The theory of alternants in the historical order of development up to 1860. *Proceed. R. Soc. Edinburgh*, xxvi. pp. 357–389.]

This, the second of our papers on the history of alternants, gives an account of twenty-seven writings more or less fully occupied with the theory. Many of them, of course, contribute only a little to the total advance made in the period.

NANSON, E. J. ($1906/_9$, $^1/_{11}$)

[On a theorem of Segar's. *Messenger of Math.*, xxxvi. pp. 77–78.]
[Question 16098. *Educ. Times*, lix. p. 496.]

Attention is here deservedly drawn to an important theorem of 1892 (*Hist.*, iv. Chap. VI), the statement of which is now essentially as follows:

$$\begin{vmatrix} \aleph_{\alpha+\alpha'} & \aleph_{\alpha+\beta'} & \aleph_{\alpha+\gamma'} & \cdots \\ \aleph_{\beta+\alpha'} & \aleph_{\beta+\beta} & \aleph_{\beta+\gamma'} & \cdots \\ \aleph_{\gamma+\alpha'} & \aleph_{\gamma+\beta'} & \aleph_{\gamma+\gamma'} & \cdots \\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \end{vmatrix}_n$$

$$= (-1)^{\frac{1}{2}n(n-1)}\, a^{n-1}\, b^{n-1} \ldots \frac{|a^\alpha b^\beta c^\gamma \ldots|\ |a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots|}{\zeta(a, b, c, \ldots)},$$

where the variables (n in number) of the aleph functions on the left are $a, b, c, \ldots$ The improved demonstration brought forward consists in evolving the left-hand member from the right-hand member by appropriately performing the multiplications implied in the latter and using repeatedly the fact that any aleph function is expressible as the quotient of an alternant by the difference-product of the variables.

The other contribution is considerably more important, being the extension of Segar's theorem to cases where the order-number m of the given determinant is no longer necessarily the same as the number of variables. The equivalent then obtained is an aggregate of $(n)_m$ terms resembling the solitary term in Segar's case. For example, when n is 4 and m is 3 the outcome is

$$\left\{\frac{a^3b^3c^3\,|a^\alpha b^\beta c^\gamma|\,|a^{\alpha'} b^{\beta'} c^{\gamma'}|}{|a^0b^1c^2|} - \frac{a^3b^3d^3\,|a^\alpha b^\beta d^\gamma|\,|a^{\alpha'} b^{\beta'} d^{\gamma'}|}{|a^0b^1d^2|} + \ldots\right\}$$

$$\div\ |a^0b^1c^2d^3|.$$

NESBITT, A. M. (1906[1]/[10])

[Question 16076. *Educ. Times*, lix. p. 455; lx. p. 81; or *Math. from Educ. Times*, (2) xii. p. 42.]

The proof given here in reply to the proposer's question is quite unsuggestive. If it had been made dependent on the fact that the 2-line minors of the first two columns have $2\Sigma x^2 + \Sigma yz$ for a common factor, the 1-to-m generalization

$$\begin{vmatrix} -2x+y+z & y^2+yz+z^2 & x^{m+2} \\ -2y+z+x & z^2+zx+x^2 & y^{m+2} \\ -2z+x+y & x^2+xy+y^2 & z^{m+2} \end{vmatrix} = \begin{vmatrix} 1 & x & x^{m+2} \\ 1 & y & y^{m+2} \\ 1 & z & z^{m+2} \end{vmatrix} (2\Sigma x^2 + \Sigma yz)$$

would have been reached. Further, it would have been seen from the case where $m = -2$, that the elements of the third column could be increased by any quantity Q without the determinant being affected.

KOSTKA, C. (1907[20]/[1])

[Bemerkungen über symmetrischen Funktionen. *Crelle's Journ.*, cxxxii. pp. 159–166.]

These so-called " remarks " are practically a working-out of the consequences arising from the peculiar intimacy of the relation between the c-functions ($\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, . . .) and the alephs ($\Sigma\alpha$, $\Sigma\alpha^2 + \Sigma\alpha\beta$, $\Sigma\alpha^3 + \Sigma\alpha^2\beta + \Sigma\alpha\beta\gamma$, . . .), and especially the joint recurrence-formula

$$c_r - c_{r-1}\aleph_1 + c_{r-2}\aleph_2 - \ldots + (-1)^r\aleph_r = 0.*$$

So far as determinants are concerned the noteworthy result is the definite formulation of the connection between Jacobi's equivalent for $|\alpha^p\beta^q\gamma^r \ldots| \div |\alpha^0\beta^1\gamma^2 \ldots|$ and Naegelsbach's, namely, that *the diagonal suffixes of the one determinant and those of the other are viewable as conjugate partitions.* Thus, to take our old example of the 7th degree (*Hist.*, iii. pp. 136, 146)

$$|\alpha^0\beta^2\gamma^5\delta^6| \div |\alpha^0\beta^1\gamma^2\delta^3|,$$

* This is equivalent to saying that

$$x^n - c_1x^{n-1} + c_2x^{n-2} - \ldots$$

and

$$x^{-n} + \aleph_1x^{-n-1} + \aleph_2x^{-n-2} + \ldots$$

are identically reciprocal.

its two equivalents are

$$\begin{vmatrix} \aleph_0 & \aleph_2 & \aleph_5 & \aleph_6 \\ \cdot & \aleph_1 & \aleph_4 & \aleph_5 \\ \cdot & \aleph_0 & \aleph_3 & \aleph_4 \\ \cdot & \cdot & \aleph_2 & \aleph_3 \end{vmatrix}, \qquad \begin{vmatrix} c_2 & c_3 & c_5 \\ c_1 & c_2 & c_4 \\ c_0 & c_1 & c_3 \end{vmatrix}$$

where 3, 3, 1 and 3, 2, 2 are conjugate partitions of 7. The actual transformation of the one determinant into the other is also effected. Of other matter brought forward it must suffice to note that the three kinds of functions printed round the tables of 1881, are not the only ones whose expansions are thence obtainable—that indeed there is a second triad, and that each function of the six is expressible with more or less facility in terms of each of the other five.

It is a little strange that when, after twenty-six years' silence, Kostka here returns to write on the subject of the connections between symmetric functions and determinants, he should say nothing regarding the work done by others during that long period. Work of this kind had been, as we have seen, very considerable in amount and not negligible in quality; and, although it is true that the workers, or almost all of them, had seemed steadily to ignore Kostka, such a defence would be a poor excuse for retaliation.

KOSTKA, C. (1907 28/6)

[Tafeln und Formeln für symmetrische Funktionen. *Jahresb. d. deutschen Math.-Verein.*, xvi. pp. 429–450.]

This is quite a valuable paper, although it contains nothing of importance that is really new. The results given in it are those of the author's papers of 1875, 1876, 1881, 1907; but they are now presented, each in its proper niche, as parts of an organic whole and as viewed by the author from his latest vantage-ground; and the exposition besides being unified is improved in points of detail. Instead of reprinting one of his eight tables for illustration purposes he generously gives a newly calculated ninth. There is no improvement, however, in the matter of references to the writings of other workers: indeed, not one of those using determinants like himself in the investigation of symmetric functions is ever mentioned.

ROSS, C. M. ($1907^1/_9$)

[Question 16276. *Educ. Times*, lx. p. 415; lxi. p. 37; or *Math. from Educ. Times*, (2) xiii. pp. 107–108; xiv. pp. 26–27.]

The purely determinantal result reached here is, after a slight correction,

$$\begin{vmatrix} 1 & \sin\alpha & \cos\alpha & \sin 2\alpha \\ 1 & \sin\beta & \cos\beta & \sin 2\beta \\ 1 & \sin\gamma & \cos\gamma & \sin 2\gamma \\ 1 & \sin\delta & \cos\delta & \sin 2\delta \end{vmatrix} = 2^5 \cos\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\cdot \Pi \sin\tfrac{1}{2}(\delta-\gamma).$$

KOSTKA, C. (1908 Easter)

[Tafeln für symmetrische Funktionen bis zur elften Dimension, mit kurzen Erläuterungen. *Sch.-Progr.*, 10 pp. + 11 Taf. Insterburg.]

Of course the important thing to be noted here is the addition (10^{th} and 11^{th}) made to the tables.

THIELE, T. N. ($1905^{12}/_1$): NÖRLUND, N. E. ($1908^{21}/_8$, $1910^{10}/_1$)

(See under this heading in Chap. XI.)

SCHUH, F. (1908)

[Vraagstuk 76. *Wiskundige Opgaven*, x. pp. 188–192.]

The subject here is the case of the double alternant

$$\left| (a_1+b_1)^p \; (a_2+b_2)^p \ldots (a_n+b_n)^p \right|$$

where $p=n$, Zehfuss (1859) having already dealt with the case where $p=n-1$ (*Hist.*, ii. p. 190). The actual object of quest is the quotient got by dividing the determinant by the difference-product of the a's and the difference-product of the b's; and this is found to be

$$(-1)^{\frac{1}{2}n(n-1)}\cdot (n)_1 (n)_2 \ldots (n)_{n-1}\cdot \sum_{i=0}^{i=n} \left\{ \Sigma a_1 a_2 \ldots a_{n-i}\cdot \Sigma b_1 b_2 \ldots b_i \right\};$$

for example, when n is 3, the determinant equals

$$-(1.3.3.1).\zeta^{\frac{1}{2}}(a_1a_2a_3)\cdot\zeta^{\frac{1}{2}}(b_1b_2b_3)$$
$$\times\left\{a_1a_2a_3+\tfrac{1}{3}\Sigma a_1a_2\Sigma b_1+\tfrac{1}{3}\Sigma a_1\Sigma b_1b_2+b_1b_2b_3\right\}.$$

KOSTKA, C. (1908)

[Zur Grundaufgabe der symmetrische Funktionen. *Schriften d. phys.-ökon. Ges.* (Königsberg i. Pr.), lix. pp. 374–384.]

Strange to say, Kostka is here again writing, not to break fresh ground, but simply to give as in the preceding year a fairly detailed account of his past work on symmetric functions. Evidently also the two accounts are written for the same class of readers—those unacquainted with the author's previous papers; and though they are clearly written and on a number of minor points are supplementary to one another, one cannot but feel that the author is lucky in the double opportunity given him.

SIBIRANI, F. (1909/5, 6)

[Sopra i polinomi trigonometrici ed un determinante relativo. *Giornale di Mat.*, xlvii. pp. 125–131.]

The determinant referred to is that which has for its rth row

$$1 \quad \sin a_r \quad \cos a_r \quad \sin 2a_r \quad \cos 2a_r \;\ldots\; \sin na_r \quad \cos na_r,$$

and which we have already seen discussed more than once, perhaps most fully by Weihrauch in 1889 (*Hist.*, iv. pp. 171, 176–177). The result is different in form from Scott's of 1879 (*Hist.*, iii. p. 168), but is not more concise.

MUIR, T. (1909¹⁵/9)

[Borchardt's form of the eliminant of two equations of the n^{th} degree. *Transac. R. Soc. S. Africa*, i. pp. 447–452.]

As a substitute for a laboriously proved result of Borchardt's the following widely general theorem is given: *If $\zeta^{\frac{1}{2}}$ be used as the functional symbol for the difference-product, and*

$$F(x,y) \quad for \quad \begin{array}{cccc|c} 1 & x & & x^{n-1} & \\ \hline a_{11} & a_{12} & \dots & a_{1n} & 1 \\ a_{21} & a_{22} & \dots & a_{2n} & y \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} & y^{n-1} \end{array}$$

then

$$\begin{aligned} \zeta^{\frac{1}{2}}(\beta_0, \beta_1, \dots, \beta_n)\cdot F(\beta_0, \beta_0) &- \zeta^{\frac{1}{2}}(\beta_0, \beta_2, \dots, \beta_n)\cdot F(\beta_0, \beta_1) \\ &+ \zeta^{\frac{1}{2}}(\beta_0, \beta_1, \beta_3, \dots, \beta_n)\cdot F(\beta_0, \beta_2) - \dots \\ &+ (-1)^n \zeta^{\frac{1}{2}}(\beta_0, \beta_1, \dots, \beta_{n-1})\cdot F(\beta_0, \beta_n) = 0. \end{aligned}$$

For proof we have only to look for the cofactor of a_{rs} on the left-hand side of the asserted equality. Now in $F(x, y)$ this cofactor is seen to be $y^{r-1}x^{s-1}$, therefore the full cofactor sought is

$$\begin{aligned} \zeta^{\frac{1}{2}}(\beta_1, \beta_2, \dots)\cdot \beta_0{}^{r-1}\beta_0{}^{s-1} &- \zeta^{\frac{1}{2}}(\beta_0, \beta_2, \dots)\cdot \beta_0{}^{r-1}\beta_1{}^{s-1} \\ &+ \zeta^{\frac{1}{2}}(\beta_0, \beta_1, \beta_3, \dots)\cdot \beta_0{}^{r-1}\beta_2{}^{s-1} - \dots \\ &+ (-1)^n\, \zeta^{\frac{1}{2}}(\beta_0, \beta_1, \dots, \beta_{n-1})\cdot \beta_0{}^{r-1}\beta_n{}^{s-1} \end{aligned}$$

which is evidently the development of

$$\beta_0{}^{r-1}\cdot(-1)^n \begin{vmatrix} 1 & \beta_0 & \beta_0{}^2 & \dots & \beta_0{}^{n-1} & \beta_0{}^{s-1} \\ 1 & \beta_1 & \beta_1{}^2 & \dots & \beta_1{}^{n-1} & \beta_1{}^{s-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \beta_n & \beta_n{}^2 & \dots & \beta_n{}^{n-1} & \beta_n{}^{s-1} \end{vmatrix}$$

arranged according to the elements of the last column of the determinant. As, however, $s \not> n$, the said last column must be identical with a preceding column: and thus the theorem is proved.

SIBIRANI, F. (1909[19]/12)

[Un determinante affine a quello di Vandermonde. *Atti . . . Istituto Veneto* . . ., lxix. pp. 447–451.]

The problem which the author here sets himself is essentially the same as Franke's of 1876 (*Hist.*, iii. p. 159) and Schendel's of 1891 (*Hist.*, iv. pp. 178–180): namely, the evaluation of a determinant the main part of whose equivalent is *a difference-product with repeated factors.* Both as regards method of solution

and clearness of exposition Sibirani's paper is an improvement. His expression for the equivalent is of necessity not shortened, and does not differ from Franke's save in the form of the sign-factor, being

$$\begin{aligned}&(-1)^{m_1m_2+(m_1+m_2)m_3+\ \dots\ +(m_1+m_2+m_3+\ \dots\ +m_{r-1})m_r}\\ &\cdot\{1!\,2!\dots(m_1-1)!\}\{1!\,2!\dots(m_2-1)!\}\dots\{1!\,2!\dots(m_r-1)!\}\\ &\cdot\left.\begin{matrix}(b_1-b_2)^{m_1m_2}(b_1-b_3)^{m_1m_3}\dots(b_1-b_r)^{m_1m_r}\\ (b_2-b_3)^{m_2m_3}\dots(b_2-b_r)^{m_2m_r}\\ \dots\dots\dots\\ (b_{r-1}-b_r)^{m_{r-1}m_r}\end{matrix}\right\}\end{aligned}$$

where $b_1, b_2, \dots, b_r$ are the variables and $m_1, m_2, \dots, m_r$ are the numbers of rows in which $b_1, b_2, \dots, b_r$ respectively occur.

LECAT, M.: AURIC, A., ETC. (1910–1911)

[Question 3788. (Quotient of a simple alternant by the difference-product of the variables.) *L'Intermédiaire des Math.*, xvii. p. 266; xviii. pp. 91–96; xxiii. 110–111.]

The four replies given deal of course more or less imperfectly with an already very widely studied subject.

SIBIRANI, F. ($1912^{11}/_4$)

[Sopra due tipi di determinanti e sopra i polinomi trigonometrici ed iperbolici pari e dispari. *Rendic. . . . Istituto Lombardo* . . ., (2) xlv. pp. 403–411.]

Of the clearly arranged series of results here brought forward the two which are fundamental are:

1. *The determinant whose matrix is the sum of the matrices of* $|a^0b^1c^2\dots|_n$ *and* $|a^0b^{-1}c^{-2}\dots|_n$ *is equal to*

$$2\,|(a+a^{-1})^0(b+b^{-1})^1(c+c^{-1})^2\dots|_n.$$

2. *The determinant whose matrix is the difference of the matrices of* $|a^1b^2c^3\dots|$ *and* $|a^{-1}b^{-2}c^{-3}\dots|$ *is equal to*

$$(a-a^{-1})(b-b^{-1})(c-c^{-1})\dots|(a+a^{-1})^0(b+b^{-1})^1(c+c^{-1})^2\dots|_n.$$

By way of proof the author falls back on Cauchy's original method of treating alternants, namely, ascertaining the substitutions

which result in the vanishing of the function under consideration. By utilizing the exponential expressions for the simple circular and hyperbolic functions he then makes the two deductions

$$\begin{aligned} &| \cos 0x_1 \cdot \cos 1x_2 \ldots \cos(n-1)x_n | \\ &\qquad = 2^{\frac{1}{2}(n-1)(n-2)} \cdot | \cos^0 x_1 \cos^1 x_2 \ldots \cos^{n-1} x_n | \end{aligned}$$

$$\begin{aligned} &| \sin x_1 \sin 2x_2 \ldots \sin nx_n | \\ &\qquad = 2^{\frac{1}{2}n(n-1)} \cdot \sin x_1 \ldots \sin x_n | \cos^0 x_1 \cos^1 x_2 \ldots \cos^{n-1} x_n | \end{aligned}$$

and other two transcribable therefrom by simply writing everywhere cosh for cos and sinh for sin. It is not noted, however, that the first two of the four were reached otherwise by Prouhet in 1857 (*Hist.*, ii. pp. 187–188).

It will be observed that the author says nothing of the determinant whose matrix is the *sum* of the matrices of $| a^1b^2c^3 \ldots |$ and $| a^{-1}b^{-2}c^{-3} \ldots |$. This omission is the more to be regretted because the trigonometrical case of it,

$$| \cos x_1 \cos 2x_2 \ldots \cos nx_n |$$

had not been previously dealt with, although proposed for evaluation by Nanson in 1903.* Taking a different mode of procedure from Sibirani, and bearing in mind that

$$\begin{aligned} &\left(a+\frac{1}{a}\right)^n \\ &\qquad = \left(a^n+\frac{1}{a^n}\right) + (n)_1\left(a^{n-1}+\frac{1}{a^{n-1}}\right) + (n)_2\left(a^{n-2}+\frac{1}{a^{n-2}}\right) + \ldots \end{aligned}$$

we have for the 4th order

$$\begin{aligned} &\left| \left(a+\frac{1}{a}\right) \left(b^2+\frac{1}{b^2}\right) \left(c^3+\frac{1}{c^3}\right) \left(d^4+\frac{1}{d^4}\right) \right| \\ &\qquad = \begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & a+\frac{1}{a} & a^2+\frac{1}{a^2} & a^3+\frac{1}{a^3} & a^4+\frac{1}{a^4} \\ 1 & b+\frac{1}{b} & b^2+\frac{1}{b^2} & b^3+\frac{1}{b^3} & b^4+\frac{1}{b^4} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \end{aligned}$$

* *Educ. Times,* lvi. p. 157.

$$= \begin{vmatrix} 1 & . & 2 & . & 6 \\ 1 & a+\frac{1}{a} & \left(a+\frac{1}{a}\right)^2 & \left(a+\frac{1}{a}\right)^3 & \left(a+\frac{1}{a}\right)^4 \\ 1 & b+\frac{1}{b} & \left(b+\frac{1}{b}\right)^2 & \left(b+\frac{1}{b}\right)^3 & \left(b+\frac{1}{b}\right)^4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \left| \alpha^0\beta^1\gamma^2\delta^3 \right| \cdot \left\{ \alpha\beta\gamma\delta + 2\Sigma\alpha\beta + 6 \right\}$$

if α, β, . . . be temporarily written for $a + a^{-1}$, $b + b^{-1}$, . . . The similar developments for the 5^{th} and 6^{th} orders are

$$\left| \alpha^0\beta^1\gamma^2\delta^3\epsilon^4 \right| \cdot \left\{ \alpha\beta\gamma\delta\epsilon + 2\Sigma\alpha\beta\gamma + 6\Sigma\alpha \right\},$$
$$\left| \alpha^0\beta^1\gamma^2\delta^3\epsilon^4\zeta^5 \right| \cdot \left\{ \alpha\beta\gamma\delta\epsilon\zeta + 2\Sigma\alpha\beta\gamma\delta + 6\Sigma\alpha\beta + 20 \right\},$$

the coefficients 2, 6, 20, . . . being $(2)_1$, $(4)_2$, $(6)_3$, . . . ; so that, as in the other trigonometrical cases,

$$\left| \cos x_1 \cdot \cos 2x_2 \cdot \ \ldots \ \cos nx_n \right|$$

is seen to be divisible by the difference-product of $\cos x_1$, $\cos x_2$, . . . , $\cos x_n$, the quotient now, however. being more complicated, namely,

$$2^{\frac{1}{2}n(n-3)} \left\{ 2^n \cos x_1 \cos x_2 \ldots \cos x_n + 2^{n-} \cdot (2)_1 \Sigma \cos x_1 \ldots \cos x_{n-2} \right.$$
$$+ 2^{n-4} \cdot (4)_2 \Sigma \cos x \ldots \cos x_{n-4}$$
$$\left. + \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \right\}.$$

ROSS, C. M. (1912¹/₉)

[Question 17366. *Educ. Times*, lxv. pp. 393, 525; or *Math. from Educ. Times*, (2) xxiii. p. 103.]

Here the result is

$$\left| \left(\frac{\partial}{\partial u_1}\right)^0 \left(\frac{\partial}{\partial u_2}\right)^1 \ldots \left(\frac{\partial}{\partial u_n}\right)^{n-1} \right| \cdot \left| u_1{}^0 u_2{}^1 \ldots u_n{}^{n-1} \right|$$
$$= n^n \cdot 1!\,2! \ldots (n-1)!,$$

for, the differentiations being performed in the quasi product-determinant, there is found only one non-vanishing term, namely,

$$n(n \cdot 1!)\,(n \cdot 2!) \ldots (n \cdot \overline{n-1}!).$$

MUIR, T. ($1912^{16}/_{10}$)

[Note on double alternants. *Transac. R. Soc. S. Africa*, iii. pp. 177–185.]

The subject here is the quotient of

$$\left| (a_1 + b_1)^p (a_2 + b_2)^p \dots (a_n + b_n)^p \right|$$

by the difference-product of the a's and the difference-product of the b's or, say,

$$D_{n:p} \div \zeta_1^{\frac{1}{2}} \zeta_2^{\frac{1}{2}}.$$

When $n = p = 3$ it is shown to be

$$\begin{vmatrix} a_1a_2a_3 + b_1b_2b_3 & \Sigma b_1b_2 & \Sigma b_1 \\ \Sigma a_1a_2 & . & -3 \\ \Sigma a_1 & -3 & . \end{vmatrix},$$

when $n = p = 4$ to be

$$\begin{vmatrix} a_1a_2a_3a_4 + b_1b_2b_3b_4 & \Sigma b_1b_2b_3 & \Sigma b_1b_2 & \Sigma b_1 \\ \Sigma a_1a_2a_3 & . & . & -4 \\ \Sigma a_1a_2 & . & -6 & . \\ \Sigma a_1 & -4 & . & . \end{vmatrix}$$

and so on generally, the main auxiliary in the two modes of proof being the multiplication-theorem. The result taken in conjunction with one of Scott's (*Hist.*, iii. pp. 168–169) gives rise to a striking relation between a determinant and a permanent; for example

$$\begin{vmatrix} a_1a_2a_3 + b_1b_2b_3 & \Sigma b_1b_2 & \Sigma b_1 \\ \Sigma a_1a_2 & . & -3 \\ \Sigma a_1 & -3 & . \end{vmatrix}$$

$$= (-1)^{\frac{1}{2}3(3-1)} \frac{(3)_1(3)_2}{3!} \begin{vmatrix} a_1 + b_1 & a_1 + b_2 & a_1 + b_3 \\ a_2 + b_1 & a_2 + b_2 & a_2 + b_3 \\ a_3 + b_1 & a_3 + b_2 & a_3 + b_3 \end{vmatrix}^{+}_{\;}.$$

The consideration of $D_{n;\,n+1}$ is next taken up, the result obtained being exemplified by

$$\frac{D_{2:3}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = \begin{vmatrix} b_1b_2 & -\Sigma b_1 & 1 & \cdot \\ -\Sigma a_1 & -3 & \cdot & 1 \\ a_1a_2 & \cdot & -3 & -\Sigma a_1 \\ \cdot & b_1b_2 & -\Sigma b_1 & a_1a_2 \end{vmatrix},$$

$$\frac{D_{3:4}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = \begin{vmatrix} -b_1b_2b_3 & \Sigma b_1b_2 & -\Sigma b_1 & 1 & \cdot \\ \Sigma a_1 & 4 & \cdot & \cdot & 1 \\ -\Sigma a_1a_2 & \cdot & 6 & \cdot & -\Sigma a_1 \\ a_1a_2a_3 & \cdot & \cdot & 4 & \Sigma a_1a_2 \\ \cdot & b_1b_2b_3 & -\Sigma b_1b_2 & \Sigma b_1 & -a_1a_2a_3 \end{vmatrix}.$$

Brief notice is then taken of $D_{n;\ n+2}$, and finally of $D_{n;\ -1}$ for the case where the two sets of variables are identical.

SHIBAYAMA, M. (1912/$_{11}$)

[A determinant, and its application to the theory of homogeneous linear differential equations. *Tôhoku Math. Journ.*, ii. pp. 143–146.]

Another rediscovery of the theorem foreshadowed by Franke in 1876 (*Hist.*, iii. p. 159) and last dealt with by Méray in 1899 (*Hist.*, iv. p. 201).

WESTENDORP, J. J. C. (1913/$_7$)

[Over eenige eigenschappen der regelmatige veelhoeken. *Wiskundig Tijdschrift*, x. pp. 15–22.]

As a preliminary there is here prefixed a simple proof that the square of the difference-product of the n^{th} roots of 1 is $(-1)^{\frac{1}{2}(n-1)(n-2)} \cdot n^n$ (*Hist.*, iii. p. 137). In the determinant form of the square in question (*Hist.*, ii. pp. 159–160) s_0 and s_n are put equal to n and each of the other s's equal to 0.

SANDERSON, M. (1913/$_{10}$)

[Formal modular invariants with application to binary modular covariants. *Transac. American Math. Soc.*, xiv. pp. 489–500.]

A section of this paper (pp. 491–493) concerns the alternant whose $(r, s)^{\text{th}}$ element is

$$a_s^{p^{(r-1)m}},$$

the particular matter discussed being its divisibility in the Galois field $[p^m]$ by another determinant differing from it only in the last row.

ANDREOLI, G. (1913)

[Nuova generalizzazione di un teorema sui determinanti di Cauchy. *Giornale di Mat.*, li. pp. 264–270.]

Apparently this was written without any knowledge of similar work later than Marcolongo's of 1887. The advance made by Weihrauch in 1889 is made a second time, but not the further step taken by Schendel in 1891. And all of them, as we have seen (*Hist.*, iv. p. 201), were alike in being unaware of Méray's work of 1865.

MATTIA, A. DE (1914/$_{1+2}$)

[Observazione su di una generalizzazione di un teorema di Stern relativo ai determinanti di Vandermonde. *Giornale di Mat.*, lii. pp. 60–62.]

For the sake of making clear the exact nature of the relationship between the new result and the so-called theorem of Stern (*Hist.*, iii. pp. 138–139) it is desirable to alter the author's mode of presentation. What is most worth noting concerns the aleph functions, and may be formulated for ourselves thus: *If the elements of* $|\alpha^0\beta^1\gamma^2\delta^3|$ *be viewed as aleph functions, and any number of additional variables be taken and inserted everywhere under the*

functional symbol, the determinant is not thereby substantially altered *: for example,

$$\begin{vmatrix} 1 & (\alpha,\ x,\ y)^1 & (\alpha,\ x,\ y)^2 & (\alpha,\ x,\ y)^3 \\ 1 & (\beta,\ x,\ y)^1 & (\beta,\ x,\ y)^2 & (\beta,\ x,\ y)^3 \\ 1 & (\gamma,\ x,\ y)^1 & (\gamma,\ x,\ y)^2 & (\gamma,\ x,\ y)^3 \\ 1 & (\delta,\ x,\ y)^1 & (\delta,\ x,\ y)^2 & (\delta,\ x,\ y)^3 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta).$$

Of course the elements on the left here may without harm be further complicated by multiplying by 1 in the form

$$\begin{vmatrix} 1 & . & . & . \\ p & 1 & . & . \\ pq & q & 1 & . \\ pqr & qr & r & 1 \end{vmatrix};$$

and this in effect is what gives us the "more general" determinant dealt with in the paper.

THEISINGER, L. (1915)

[Bemerkung über die harmonische Reihe. *Monatshefte f. Math. u. Phys.*, xxvi. pp. 132–134.]

The result obtained here with some trouble by solving a set of linear equations is the equality

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} = \frac{1}{1!\ 2!\ \ldots\ n!} \left| 1^0 2^2 3^3 \ldots n^n \right|.$$

* This may be readily proved by multiplying the left-hand member by

$$\begin{vmatrix} 1 & . & . & . \\ -(x+y) & 1 & . & . \\ xy & -(x+y) & 1 & . \\ . & xy & -(x+y) & 1 \end{vmatrix}$$

or the right-hand member by

$$\begin{vmatrix} 1 & . & . & . \\ x+y & 1 & . & . \\ x^2+xy+y^2 & x+y & 1 & . \\ x^3+\ldots+y^3 & x^2+xy+y^2 & x+y & 1 \end{vmatrix}.$$

A similar result is that

$$\begin{vmatrix} 1 & (\alpha+x+\ldots)^1 & (\alpha+x+\ldots)^2 & (\alpha+x+\ldots)^3 \\ 1 & (\beta+x+\ldots)^1 & (\beta+x+\ldots)^2 & (\beta+x+\ldots)^3 \\ 1 & (\gamma+x+\ldots)^1 & (\gamma+x+\ldots)^2 & (\gamma+x+\ldots)^3 \\ 1 & (\delta+x+\ldots)^1 & (\delta+x+\ldots)^2 & (\delta+x+\ldots)^3 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha,\beta,\gamma,\delta).$$

Knowing however, as we do (*Hist.*, ii. pp. 175–176), that

$$\begin{aligned} \left| 1^0 2^2 3^3 \ldots n^n \right| &= \left| 1^0 2^1 3^2 \ldots n^{n-1} \right| \cdot \Sigma \\ &= (n-1)!\,(n-2)! \ldots 2!\; 1! \cdot \Sigma \end{aligned}$$

where Σ is the sum of the products of 1, 2, . . . , n taken $n-1$ at a time, we have the right-hand side of the equality

$$\begin{aligned} &= \frac{\Sigma}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}, \end{aligned}$$

as desired.

MUIR, T. (1915[1]/[6])

[Question 18015. *Educ. Times*, lxviii. p. 238; *Math. from Educ. Times*, (2) xxix. pp. 58–60; *Math. Quest. and Sol.*, i. p. 16.]

The new point of interest here in regard to Borchardt's determinant of 1855 (*Hist.*, ii. pp. 173–175) is that by following a different mode of evaluation there emerges a curious equality between a permanent and a determinant, namely,

$$\overset{+}{\left|\begin{matrix} (a_1+b_2)(a_1+b_3) & (a_1+b_3)(a_1+b_1) & (a_1+b_1)(a_1+b_2) \\ (a_2+b_2)(a_2+b_3) & (a_2+b_3)(a_2+b_1) & (a_2+b_1)(a_2+b_2) \\ (a_3+b_2)(a_3+b_3) & (a_3+b_3)(a_3+b_1) & (a_3+b_1)(a_3+b_2) \end{matrix}\right|^{+}}$$

$$= \left|\begin{matrix} 3 & 2\Sigma a_1 + \Sigma b_1 & \Sigma b_1 b_2 - \Sigma a_1{}^2 - \Sigma a_1 a_2 \\ \Sigma b_1 & -2\Sigma a_1 a_2 & \Sigma a_1{}^2 a_2 + 2(a_1 a_2 a_3 + b_1 b_2 b_3) \\ \Sigma b_1 b_2 & 2a_1 a_2 a_3 - b_1 b_2 b_3 & -\Sigma a_1{}^2 a_2 a_3 \end{matrix}\right|.$$

ROHN, K. (1915[19]/[7])

[Auswerthung einer Determinante. *Berichte* . . . *Ges. d. Wiss.*, (Leipzig), lxvii. pp. 298–302.]

The determinant in question is a double alternant not hitherto evaluated or even brought to notice, namely,

$$\left| \frac{y_r + x_s}{y_r - x_s} \right|_n, \quad \text{or } R_n \text{ say.}$$

The tedious process of removing its factors may perhaps be best arranged as follows, the case being taken where n is 4. Performing

$$\mathrm{col}_1 - \mathrm{col}_2, \quad \mathrm{col}_2 - \mathrm{col}_3, \quad \mathrm{col}_3 - \mathrm{col}_4$$

we remove $2(x_1 - x_2)$, $2(x_2 - x_3)$, $2(x_3 - x_4)$: performing on the resulting determinant $\mathrm{col}_1 - \mathrm{col}_2$, $\mathrm{col}_2 - \mathrm{col}_3$ we remove $x_3 - x_1$, $x_4 - x_2$: and then performing $\mathrm{col}_1 - \mathrm{col}_2$ we remove $x_4 - x_1$: in all there has thus been removed the difference-product of the x's and the factor -2^3, that is

$$-2^3\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4).$$

Next by removing the reciprocal of the product of all the denominators of the original elements, say the product $\Pi_{4,4}$, we find remaining a determinant readily simplifiable without removing factors, the full result now being

$$-2^3\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4) \cdot \frac{1}{\Pi_{4,4}} \cdot \begin{vmatrix} y_1 & y_1^2 & y_1^3 & y_1^4 - x_1x_2x_3x_4 \\ y_2 & y_2^2 & y_2^3 & y_2^4 - x_1x_2x_3x_4 \\ y_3 & y_3^2 & y_3^3 & y_3^4 - x_1x_2x_3x_4 \\ y_4 & y_4^2 & y_4^3 & y_4^4 - x_1x_2x_3x_4 \end{vmatrix}.$$

Lastly, by partitionment of the determinant thus arrived at into two, it is easily seen to be equal to

$$\zeta^{\frac{1}{2}}(y_1, y_2, y_3, y_4) \cdot (y_1y_2y_3y_4 + x_1x_2x_3x_4);$$

so that for our final result we have

$$R_4 = -2^{4-1}(x_1x_2x_3x_4 + y_1y_2y_3y_4) \cdot \frac{\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4) \cdot \zeta^{\frac{1}{2}}(y_1, y_2, y_3, y_4)}{\Pi_{4\cdot 4}}.$$

We note for ourselves further that since Cauchy's original double-alternant, C_n say, which differs from R_n merely in having 1 for the numerator of every element, is (*Hist.*, i. p. 345) equal to

$$(-1)^{\frac{1}{2}n(n-1)} \cdot \frac{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}}{\Pi_{n\cdot n}},$$

we consequently obtain the interesting connecting result

$$R_n = (-1)^{\frac{1}{2}(n+1)(n+2)} \cdot (x_1x_2 \ldots x_n + y_1y_2 \ldots y_n) \cdot C_n.$$

Rohn's procedure consists in giving a direct proof of this, and then introducing Cauchy's expression for C_n. He also makes an interesting deduction by changing all the y's into their reciprocals.

MUIR, T. ($1915^{28}/_{12}$)

(See under this heading in Chapter on Skew Determinants.)

MUIR, T. ($1916^{17}/_{4}$)

[A class of alternants with trigonometrical elements. *S. African Journ. of Sci.*, xiii. pp. 197–200.]

Here there is first established the identity

$$A \sin\alpha + B \sin 3\alpha + F \cos\alpha + G \cos 3\alpha = \sin 5\alpha,$$

where A, B, F, G are symmetric functions of α, β, γ, δ, namely

$$\begin{aligned} A &= -\Sigma\cos(2\alpha+2\beta) - \Sigma\cos(2\alpha+2\beta+2\gamma), \\ F &= -\Sigma\sin(2\alpha+2\beta) + \Sigma\sin(2\alpha+2\beta+2\gamma), \\ B &= \quad \Sigma\cos 2\alpha + \cos\Sigma 2\alpha, \\ G &= \quad \Sigma\sin 2\alpha - \sin\Sigma 2\alpha. \end{aligned}$$

The partial symmetry of the identity is then taken advantage of, α being interchanged with each of the other angles, with the result that there is thus provided a set of four equalities viewable as linear equations for determining A, B, F, G. The ultimate outcome is the evaluation of eleven alternants,

$$\begin{vmatrix} \sin 3\alpha & \sin 5\alpha & \cos\alpha & \cos 3\alpha \\ \sin 3\beta & \sin 5\beta & \cos\beta & \cos 3\beta \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 16A\,\Pi\sin(\delta-\gamma),$$

$$\begin{vmatrix} \sin\alpha & \sin 5\alpha & \cos\alpha & \cos 3\alpha \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = -16B\,\Pi\sin(\delta-\gamma).$$

and so on. From the corresponding identity for the case of six angles there is deducible in like manner the evaluation of fifteen alternants of the sixth order. The case where the number of angles is odd is also briefly dealt with.

MUIR, T. ($1916^{1}/_{7}$)

[Question 18249. *Math. Quest. and Sol.*, ii. p. i; ii. pp. 57–59.]

The result established here is that each of the elementary trigonometrical functions of the sum of four angles (α, β, γ, δ)

can be expressed as the quotient of two alternants: for example,

$$\sin(\alpha+\beta+\gamma+\delta)$$
$$= \begin{vmatrix} 1 & \cos 2\alpha & \cos 4\alpha & \sin 2\alpha \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \div 2 \begin{vmatrix} \sin\alpha, & \sin 3\alpha, & \cos\alpha, & \cos 3\alpha \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The mode of verification used does not suggest a generalization.

CORPUT, J. G. v. d. (1916)

[Vraagstuk 76. *Wiskundige Opgaven*, xii. pp. 176–177.]

The main result here is not essentially different from the representation of an aleph function as the quotient of an alternant by the difference-product (*Hist.*, i. pp. 340–342.)

MUIR, T. (1916$^5/_{11}$)

[Note on a determinant whose elements are aleph functions. *Messenger of Math.*, xlvi. pp. 108–110.]

The purpose here is exactly that of Nanson's paper of 1906, and the proof given may be described as the resolution of the left-hand member into the factors specified on the right. This is effected by substituting for every aleph function its equivalent as the quotient of an alternant by the difference-product of the variables, and then observing that the determinant of the said aleph functions is a product-determinant. For example, in the case of the 3rd order, the determinant for resolution is first changed into

$$\begin{vmatrix} |a^0b^1c^{2+\alpha+\alpha'}| & |a^0b^1c^{2+\alpha+\beta'}| & |a^0b^1c^{2+\alpha+\gamma'}| \\ |a^0b^1c^{2+\beta+\alpha'}| & |a^0b^1c^{2+\beta+\beta'}| & |a^0b^1c^{2+\beta+\gamma'}| \\ |a^0b^1c^{2+\gamma+\alpha'}| & |a^0b^1c^{2+\gamma+\beta'}| & |a^0b^1c^{2+\gamma+\gamma'}| \end{vmatrix} \div \zeta^{3}_{2},$$

and then into

$$\begin{vmatrix} a^\alpha(c-b) & b^\alpha(a-c) & c^\alpha(b-a) \\ a^\beta(c-b) & b^\beta(a-c) & c^\beta(b-a) \\ a^\gamma(c-b) & b^\gamma(a-c) & c^\gamma(b-a) \end{vmatrix} \cdot \begin{vmatrix} a^{2+\alpha'} & b^{2+\alpha'} & c^{2+\alpha'} \\ a^{2+\beta'} & b^{2+\beta'} & c^{2+\beta'} \\ a^{2+\gamma} & b^{2+\gamma'} & c^{2+\gamma'} \end{vmatrix} \div \zeta^{3}_{2},$$

which clearly

$$= - \left| a^\alpha b^\beta c^\gamma \right| \cdot \zeta^{\frac{1}{2}} \cdot a^2b^2c^2 \cdot \left| a^{\alpha'} b^{\beta'} c^{\gamma'} \right| \div \zeta^{\frac{3}{2}}$$

$$= - \quad a^2b^2c^2 \frac{\left| a^\alpha b^\beta c^\gamma \right| \cdot \left| a^{\alpha'} b^{\beta'} c^{\gamma'} \right|}{\zeta}$$

as required.

A simple-looking alternative form of the theorem is next noted, namely,

$$\begin{vmatrix} \aleph_{\alpha+\alpha'} & \aleph_{\alpha+\beta'} & \aleph_{\alpha+\gamma'} & \cdots \\ \aleph_{\beta+\alpha'} & \aleph_{\beta+\beta'} & \aleph_{\beta+\gamma'} & \cdots \\ \aleph_{\gamma+\alpha'} & \aleph_{\gamma+\beta'} & \aleph_{\gamma+\gamma'} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix}$$

$$= \begin{vmatrix} \aleph_\alpha & \aleph_\beta & \aleph_\gamma & \cdots \\ \aleph_{\alpha+1} & \aleph_{\beta+1} & \aleph_{\gamma+1} & \cdots \\ \aleph_{\alpha+2} & \aleph_{\beta+2} & \aleph_{\gamma+2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} \cdot \begin{vmatrix} \aleph_{\alpha'} & \aleph_{\beta'} & \aleph_{\gamma'} & \cdots \\ \aleph_{\alpha'-1} & \aleph_{\beta'-1} & \aleph_{\gamma'-1} & \cdots \\ \aleph_{\alpha'-2} & \aleph_{\beta'-2} & \aleph_{\gamma'-2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix}$$

and the interesting special case where $\alpha', \beta', \gamma', \ldots = 0, 1, 2, \ldots$

We may now add that had we begun with Nanson's more general determinant the two factors obtained would have been m-by-n arrays, thus giving rise to the $(n)_m$ terms of the required aggregate.

ROSS, C. M. ($1917^1/_1$)

[Questions 18359, 18364. *Math. Quest. and Sol.*, iii. pp. i-ii; iv. pp. 61-62, 93-94.]

The first result here,

$$\begin{vmatrix} (a+x)(a-y)(a-z) & (a+y)(a-z)(a-x) & (a+z)(a-x)(a-y) \\ (b+x)(b-y)(b-z) & (b+y)(b-z)(b-x) & (b+z)(b-x)(b-y) \\ (c+x)(c-y)(c-z) & (c+y)(c-z)(c-x) & (c+z)(c-x)(c-y) \end{vmatrix}$$
$$= -4(abc+xyz)(x-y)(y-z)(z-x)(a-b)(b-c)(c-a),$$

is of much greater interest than the solution given would imply. It is all-important to note that the determinant resolves itself into

$$\begin{Vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & x-y-z & yz-zx-xy & xyz \\ 1 & y-z-x & zx-xy-yz & xyz \\ 1 & z-x-y & xy-yz-zx & xyz \end{Vmatrix}$$

which, when expressed as a sum of products, leads at once to

$$-4\,|a^0b^1c^2|\cdot|x^0y^1z^2|\,(abc+xyz).$$

The null effect of the interchange of a, b, c and x, y, z is also worth noting in all but the final form.

The subject of the second question is the set of equations dealt with by Binet in 1837 (*Hist.*, ii. pp. 156–158).

KOSTKA, C. (1917/7)

[Schlussformel zur Hauptaufgabe der symmetrischen Funktionen. *Crelle's Journ.*, cxlviii. pp. 88–99.]

The first section of this paper (pp. 88–92) is practically a rehearsal of previously obtained results—a rehearsal probably deemed necessary by the writer after a silence of nine years on the subject. The other section is somewhat of the nature of a rejoinder to a dissertation written earlier in the same year by G. Junge,* and records an attempt to give a final solution to the problem of expressing $\Sigma\alpha^u\beta^v\gamma^w$ . . . in terms of the c's ($\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, . . .). To this end a formula is laboriously worked out for the final expansion of Naegelsbach's determinant of c's, namely, the determinant

$$\begin{vmatrix} c_p & c_{p+1} & c_{p+2} & \cdots \\ c_{q-1} & c_q & c_{q+1} & \cdots \\ c_{r-2} & c_{r-1} & c_r & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \quad \text{or } N(p, q, r, \ldots) \text{ say.}$$

It must suffice to say merely that the character of the result is similar to that obtained for $N(1, 1, 1, \ldots)$ by Brioschi in 1858 (*Hist.*, iii. pp. 208–209).

CALDARERA, F. (1918 20/1)

[Su taluni determinanti di forme singolari. *Atti della Accad. Gioenia* . . . (Catania) (5) xi. 17 pp.]

Almost quite three-quarters of this longish paper are occupied directly or indirectly with alternants. The author begins at the

* Zur Hauptaufgabe der symmetrischen Funktionen. Dissert. Berlin, 1917.

beginning and spares neither pains nor space to ensure a clear exposition. A number of the paragraphs at the commencement would have been a useful addition to his textbook of 1913, in which the subject hardly appears. We thus find ourselves in § 3 before we meet a determinant calling for notice. The first is the n-line determinant whose $(r, s)^{\text{th}}$ element is $\left(\dfrac{a}{2s-1}\right)^{2r} - 1$. From each row of this an evident binomial factor is removed with the almost immediate result

$$\left| \left(\frac{a}{2s-1}\right)^{2r} - 1 \right|_n$$
$$= (-1)^n \cdot a^{n(n-1)} \cdot \left(1 - \frac{a^2}{1^2}\right)\left(1 - \frac{a^2}{3^2}\right) \cdots \left(1 - \frac{a^2}{(2n-1)^2}\right)$$
$$\cdot \zeta^{\frac{1}{2}}\left(\frac{1}{1^2}, \frac{1}{3^2}, \ldots, \frac{1}{(2n-1)^2}\right),$$

which is carried a stage farther by showing that the difference-product equals

$$(-1)^{\frac{1}{2}n(n-1)} \cdot 2^{n(n-1)} \cdot \frac{2^{n-1}4^{n-2}6^{n-3} \ldots (2n-4)^2(2n-2)^1}{3^n 5^{n+1} 7^{n+2} \ldots (2n-3)^{2n-3}(2n-1)^{2n-2}}.$$

Similarly there is found

$$\left| \left(\frac{\beta}{s}\right)^{2r} - 1 \right|_n$$
$$= (-1)^n \cdot \beta^{n(n-1)} \cdot \left(1 - \frac{\beta^2}{1^2}\right)\left(1 - \frac{\beta^2}{2^2}\right) \cdots \left(1 - \frac{\beta^2}{n^2}\right)$$
$$\cdot \zeta^{\frac{1}{2}}\left(\frac{1}{1^2}, \frac{1}{2^2}, \ldots, \frac{1}{n^2}\right),$$

the difference-product in which being again partially evaluated. Next (§ 6) a set of linear equations is taken up, which leads to the evaluation of the n-line minors of the n-by-$(n+1)$ array

$$\begin{array}{cccccc}
1 & 1 & 1 & \ldots & 1 & 1 \\[4pt]
\dfrac{1}{1^2} & \dfrac{1}{2^2} & \dfrac{1}{3^2} & \ldots & \dfrac{1}{n^2} & \dfrac{1}{d^2} \\[8pt]
\dfrac{1}{1^4} & \dfrac{1}{2^4} & \dfrac{1}{3^4} & \ldots & \dfrac{1}{n^4} & \dfrac{1}{d^4} \\[4pt]
\cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot
\end{array}$$

and finally (§§ 7–9) alternants of infinite order are considered, the examples discussed being taken from the foregoing.

KOSTKA, C. (1918, 1919)

[Determinanten und symmetrische Funktionen. *Jahresb. d. deutschen Math.-Verein*, xxvii. pp. 161–165; xxviii. pp. 66–68.]

[Symmetrische Funktionen in Verbindung mit Determinanten. *Nova Acta . . . Akad. d. Naturf.*, (Halle), civ. pp. 217–304.]

These are the papers referred to at the close of our introduction to this chapter. In the last, the memoir of 1919, are embodied, of set purpose, practically all the results of the author's previous work on the subject. Further this has not been done after the fashion of a mere chronicle or compilation. The memoir is an entirely fresh exposition, consisting of a short historical introduction and five carefully planned chapters. The writer, being one who could profit by a critical study of his own past work, has used his wider and clearer vision to present the old truths in improved form. And, of course, gain is also got by the mere employment of a uniform system of notation throughout. As a consequence it is to-day the best single work on the interrelations of symmetric functions and determinants.

Its one defect—not a trivial one—lies in the fact that the contributions of all other workers in the same field are ignored. In his historical sketch, it is true, he gives a lengthy string of writers belonging to the days before symmetric functions found salvation in the union with determinants, but he practically insists that since then he has been the only saviour. In explanation of this attitude it would be wrong to suggest retaliation, as we have seemed to do above: he himself says, simply enough, "Mitarbeiter auf dem Wege sind dem Verfasser nicht bekannt geworden: daher wird nur bei einzelnen Beispielen auf frühere Behandlung hinzuweisen sein".

ARWIN, A. (1919)

[En determinantberäkning. *Mat. Tidsskrift*, B, i. (1919), pp. 70–74.]

The matter dealt with here is the solution of a set of equations whose determinant is the conjugate of the difference-product alternant—that is to say, the set associated with the name of Lagrange and solved determinantally for the first time by Cauchy (*Hist.*, ii. pp. 154–155). The solution may be compared with Grunert's of 1847.

PAULLI, H. ($1919^2/_6$)

[Bemærkninger til Opgave 375. *Mat. Tidsskrift*, A, i. pp. 74–76.]

The determinant giving rise to this is that which we have dated back to Salmon and the year 1866 (*Hist.*, iii. p. 140), and which is here viewed as included in

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin X \\ \sin\beta & \cos\beta & \sin Y \\ \sin\gamma & \cos\gamma & \sin Z \end{vmatrix},$$

a number of third columns dependent on α, β, γ being taken in succession and evaluations made in product form similar to the original: for example, when

$$X,\ Y,\ Z = \tfrac{1}{2}(2\alpha - \beta - \gamma),\ \tfrac{1}{2}(2\beta - \gamma - \alpha),\ \tfrac{1}{2}(2\gamma - \alpha - \beta)$$

the result of the evaluation is

$$16 \sin\frac{\alpha-\beta}{4} \sin\frac{\beta-\gamma}{4} \sin\frac{\gamma-\alpha}{4}$$
$$\cdot \sin\frac{2\alpha-\beta-\gamma}{4} \sin\frac{2\beta-\gamma-\alpha}{4} \sin\frac{2\gamma-\alpha-\beta}{4}$$

CHAPTER VII

Compound Determinants from 1900 to 1920

The number of writings on compound determinants in the period 1900–1920 is almost identical with the number for the immediately preceding twenty-year period. The languages used by the writers are at least as numerous: but the contributions made by the various nationalities are relatively much altered in amount. The former strikingly high proportion of Magyar work is not at all maintained, and German has somewhat fallen off: on the other hand, English and Italian have increased. Although statements at variance with historical accuracy, and the publication of old results as new, are still more common than they ought to be, pleasing evidence is not awanting of improvement in these respects. There is nothing so glaring, for example, as the fact chronicled against the writers of the two preceding periods that they had unwittingly rediscovered ten times a theorem that had appeared in so well known a journal as Liouville's in 1851.

RAHUSEN, A. E. (1889)

(See under this heading in the chapter on Orthogonants.)

HATZIDAKIS, N. J. ($1899/_7$): ARANY, D. ($1899/_5$)

[Displacements depending on one, two, . . . , k parameters in space of n dimensions. *American Journ. of Math.*, xxii. pp. 154–184; in partic. p. 176.]

[Question 36. *Math. és Phys. Lapok*, viii. p. 270.]

The theorem formulated by Hatzidakis is Zehfuss' case of Bazin's theorem of 1851 (*Hist.*, iii. p. 176); and that proposed for proof by Arany is Hunyady's of 1879 (*Hist.*, iii. pp. 205–206).

NANSON, E. J. (1900)

(See under this heading on p. 76 of *Hist.*, iv.)

CARLINI, L. (1900/$_{12}$)

[Sul prodotto di due matrici rettangolari conjugate. *Periodico di Mat.*, (2) iii. pp. 193–198; iv. pp. 175–179.]

The type of compound determinant here considered is that in which the rows are in part taken from a compound of $|a_{1n}|$ and in part from the complementary compound, the points of inquiry being as to the power of $|a_{1n}|$ which is contained in it as a factor, and as to the form of the cofactor of this power. The author writes apparently without any knowledge of Hunyady's paper of 1880 on the subject (*Hist.*, iv. pp. 204–205).

VOGT, H. (1901^1/$_5$)

[Théorème relatif aux mineurs d'un déterminant. *Nouv. Annales de Math.*, (4) i. pp. 211–214.]

The theorem in question, though here spoken of in connection with Netto, is in reality Sylvester's of March, 1851 (*Hist.*, ii. pp. 193–194). It is most easily remembered as the extensional of the futile-looking identity

$$\begin{vmatrix} a_1 & a_2 & \dots \\ b_1 & b_2 & \dots \\ \cdot & \cdot & \cdot \end{vmatrix} = |a_1 b_2 \dots|.$$

The fresh proof which Vogt submits is meant as an alternative to Netto's of 1893 (*Hist.*, iv. p. 216).

ZERR, G. B. M. (1902/$_{11}$)

[Miscellaneous Question 116. *American Math. Monthly*, viii. p. 180; ix. pp. 268–269.]

The question is avowedly carried over by the proposer from Muir's textbook (p. 194). Zerr's proof is ample and suitable, but note should be taken that the equality sought to be established

is shown elsewhere by Muir to be a case of Sylvester's of 1851 (*Hist.*, iv. pp. 206–207).

ORLANDO, L. (1902)

[Relazioni fra i minori d'ordine *s* d'un determinante di caratteristica *p*. 4 pp. Messina.]

[Relazioni fra i minori d'ordine *p* d'una matrice quadrata di caratteristica *p*. *Giornale di Mat.*, xl. pp. 275–277.]

The first of these publications is of a very unusual kind. The text, which extends to only twenty-four (24) lines. occupies the third of the four unnumbered pages: the first page is like the title-page of an ordinary book: the second page contains only two words intimating copyright, and a foot-line giving the name of the printing firm and the date " 10th August, 1902 ": and the fourth page is entirely blank. The communication to the *Giornale* appeared in September, but, from a comparison of the titles, it seems not unlikely that it was the first written of the two. In any case, it is the theorem of the leaflet, as being the more general, that we have to record. In effect it is: *If the non-zero rank of* $|\mathrm{a}_{1n}|$ *be* k, *then the non-zero rank of the* m^{th} *compound of* $|\mathrm{a}_{1n}|$ *is* $(\mathrm{k})_m$. The main part of the proof rests on the simple fact that any minor of a compound of $|a_{1n}|$ is a compound of a minor of $|a_{1n}|$, and therefore a power of this minor. In the case of the less general theorem a different proof is given.

NICOLETTI, O. ($1902^{12}/_6$)

(See under this heading in Chapter I.)

STUYVAERT, M. ($1903^1/_8$)

[Question 1429. *Mathesis*, (3) iii. p. 184; xxxvi. pp. 64–65.]

The equality here proposed for proof concerns the 4-by-6 array

$$\left\|\begin{matrix} a & b & c & d & & \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\|,$$

being

$$\Big|\, |b_4c_5d_6| \quad |a_1c_5d_6| \quad |a_1b_2d_6| \quad |a_1b_2c_3| \,\Big|$$
$$= |a_1b_4c_5d_6| \cdot |a_1b_2c_5d_6| \cdot |a_1b_2c_3d_6|.$$

It closely resembles Metzler's of 1897 (*Hist.*, iv. p. 221), but both are cases of Reiss' of 1867 (*Hist.*, iii. p. 189).

BAKER, H. F. ($1904/_4$)

[Note on Sylvester's theorems on determinants. *Collected Math. Papers of J. J. Sylvester*, i. pp. 647–650.]

The theorems thus vaguely referred to are those brought forward in Sylvester's papers of March and August 1851 (*Hist.*, ii. pp. 58–61, 61–62, 193–197). The one which at present concerns us is the last of the three, being that now known as the extensional of the main property of the k^{th} compound of $|a_{1n}|$,—that is to say, the extensional of the equality

$$\Big|\, |a_{1n}| \,\Big|_k = |a_{1n}|^{(n-1)_{k-1}}.$$

The exact character of the proof given by Baker will be better understood if we recall the fact that to establish the extensional of any equality E we may either simply use the Law of Extensible Minors and so attain our end at one stroke, or we may, for the special case of E, follow at full length the procedure by which the said Law itself in all its generality was established—that is to say, employ on E in the appropriate manner the fundamental properties of the adjugate. Now Baker's procedure is neither of these, although necessarily akin to the second. It differs from the latter in the first place by beginning with the adjugate and not calling in E till a late stage; and, in the second place, in using the adjugate in the variant form of the reciprocal or " inverse ". In illustration of it let us present in our own way the case where n is 4 (the initial determinant being $|a_1b_2c_3d_4|$), k is 2, and the extension is $e_5f_6g_7$: and where, therefore,

$$\Big|\, |a_1b_2c_3d_4| \,\Big|_2 = |a_1b_2c_3d_4|^3,$$

and the compound determinant to be considered is

$$\begin{vmatrix} |a_1b_2e_5f_6g_7| & |a_1b_3e_5f_6g_7| & \cdots & |a_3b_4e_5f_6g_7| \\ |a_1c_2e_5f_6g_7| & |a_1c_3e_5f_6g_7| & \cdots & |a_3c_4e_5f_6g_7| \\ \cdot & \cdot & \cdots & \cdot \\ |c_1d_2e_5f_6g_7| & |c_1d_3e_5f_6g_7| & \cdots & |c_3d_4e_5f_6g_7| \end{vmatrix}, \text{ or } X \text{ say.}$$

Denoting $|a_1b_2c_3d_4e_5f_6g_7|$ by Δ and its adjugate by $|A_1B_2\ldots G_7|$ we have

$$X = \begin{vmatrix} |C_3D_4|\Delta^{-1} & |C_2D_4|\Delta^{-1} & \ldots & |C_1D_2|\Delta^{-1} \\ |B_3D_4|\Delta^{-1} & |B_2D_4|\Delta^{-1} & \ldots & |B_1D_2|\Delta^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ |A_3B_4|\Delta^{-1} & |A_2B_4|\Delta^{-1} & \ldots & |A_1B_2|\Delta^{-1} \end{vmatrix}$$

$$= \Delta^{-6} \cdot \text{second compound of } |A_1B_2C_3D_4|$$

$$= \Delta^{-6}\,|A_1B_2C_3D_4|^3 = \Delta^{-6}\{\Delta^3 \cdot |e_5f_6g_7|\}^3$$

$$= |a_1b_2c_3d_4e_5f_6g_7|^3 \cdot |e_5f_6g_7|^3,$$

as required, and as already given as an illustration on p. 196 of *Hist.*, ii.

DICKSON, L. E. ($1905^1/_1$)

[Expressions for the elements of a determinant in terms of the minors of a given order. Generalization of a theorem of Studnička. *American Math. Monthly*, xii. pp. 217–221.]

This is the first direct attempt to find expressions of the kind in question: anything cognate of earlier date being of little consequence (*Hist.*, iv. p. 371). To begin with, a solution involving any unnecessary variety of radicals is deprecated: it being indeed affirmed before proof begins that the a's of $|a_{1n}|$ are expressible in terms of the k-line minors of $|a_{1n}|$ without using any irrationality save the k^{th} root of a rational function of the said minors. One particular case is first solved, not merely on its own account but in order to be used as a base for the solution of the others, namely, the case where $k = n - 1$; in other words, where it is the primary minors that are given. The case where $k = 2$ and the case where $k = 3$ are then dealt with in detail in order to indicate the general procedure, the first stage of which when $k = 2$ is to find 3-by-3 of the elements from knowing the adjugate elements, and when $k = 3$ to find 4-by-4 of the unknown elements from similar data. With the help of these the remaining $n^2 - (k+1)^2$ elements are obtained in instalments, the operations, roughly speaking, being substitutions for the newly-found elements and solutions of sets of linear equations so obtained.

An alternative mode of solution is also given for the case

where $k = 2$, the basis of which is Hermite's so-called condensation theorem of 1849 (*Hist.*, ii. p. 46), although not recognized as such. In connection therewith it is unfortunately also necessary to note that the theorem spoken of in the title as Studnička's is wrongly so called, and that even the wide generalization of it effected by Dickson was already known in the early part of 1879 before Studnička's paper had appeared.

MASCHKE, H. (1905$^{22}/_{4}$)

[Differential parameters of the first order. *Transac. American Math. Soc.*, vii. pp. 69–80, 571.]

In the introduction to this paper attention is drawn to four theorems in determinants, and the first section (pp. 69–72) is devoted to the formal enunciation and proof of them. As a result of this segregation, it is all the more disappointing to find that every one of the four is a rediscovery. Only one, it is true, belongs strictly to our present chapter—Bazin's important and oft-discovered * theorem of 1851 (*Hist.*, ii. pp. 206–208); but it is as well to point out once and for all that the first of the four is Sylvester's equally old expansion-theorem (*Hist.*, ii. pp. 61–62), and that the third is Casorati's theorem of 1874 on Jacobians (*Hist.*, iii. p. 269).

MUIR, T. (1905$^{6}/_{11}$)

[Elimination in the case of equality of fractions whose numerators and denominators are linear functions of the variables. *Transac. R. Soc. Edinburgh*, xlv. pp. 1–7.]

The procedure followed here will be readily guessed from observing the lemma established at the outset, namely: *If we have given*

$$\frac{a_1x_1 + b_1x_2 + \ldots + l_1x_n}{\alpha_1x_1 + \beta_1x_2 + \ldots + \lambda_1x_n} = \ldots = \frac{a_nx_1 + b_nx_2 + \ldots + l_nx_n}{\alpha_nx_1 + \beta_nx_2 + \ldots + \lambda_nx_n}$$
$$= \frac{1}{r},$$

* E.g. six times in the period 1860–1880.

then

$$\left| a_1 - r\alpha_1 \quad b_2 - r\beta_2 \quad \ldots \quad l_n - r\lambda_n \right| = 0.$$

An additional equal fraction being given us we have at once the theorem; *the eliminant of the set of equations*

$$\frac{a_1x_1 + b_1x_2 + \ldots + l_1x_n}{\alpha_1x_1 + \beta_1x_2 + \ldots + \lambda_1x_n} = \ldots = \frac{a_{n+1}x_1 + b_{n+1}x_2 + \ldots + l_{n+1}x_n}{\alpha_{n+1}x_1 + \beta_{n+1}x_2 + \ldots + \lambda_{n+1}x_n}$$

is

$$\begin{vmatrix} D_1 & \Sigma D'_1 & \Sigma D''_1 & \ldots \\ D_2 & \Sigma D'_2 & \Sigma D''_2 & \ldots \\ \cdot & \cdot & \cdot & \cdot \\ D_{n+1} & \Sigma D'_{n+1} & \Sigma D''_{n+1} & \ldots \end{vmatrix}$$

where

$$D_1 = |a_1b_2 \ldots l_n|, \quad D_2 = |a_2b_3 \ldots l_{n+1}|, \ldots, \quad D_{n+1} = |a_{n+1}b_1 \ldots l_{n-1}|,$$

and where D'_r *indicates that any one of the letters of* D_r *has been replaced by the corresponding letter of the other alphabet,* D''_r *that any two letters have been similarly treated, and so on.* By use of a similar process an alternative theorem is next got, namely: *The eliminant of the set of equations*

$$\frac{a_1x_1 + b_1x_2 + \ldots + l_1x_n}{\alpha_1x_1 + \beta_1x_2 + \ldots + \lambda_1x_n} = \ldots = \frac{a_{n+1}x_1 + b_{n+1}x_2 + \ldots + l_{n+1}x_n}{\alpha_{n+1}x_1 + \beta_{n+1}x_2 + \ldots + \lambda_{n+1}x_n}$$

is

$$\begin{vmatrix} D_1 & \Sigma D'_1 & \Sigma D''_1 & \ldots \\ D_2 & \Sigma D'_2 & \Sigma D''_2 & \ldots \\ \cdot & \cdot & \cdot & \cdot \\ D_n & \Sigma D'_n & \Sigma D''_n & \ldots \end{vmatrix} = 0,$$

where

$$D_1 = |a_1b_2c_3 \ \ldots \ l_n\alpha_{n+1}|, \quad D_2 = |a_1b_2c_3 \ \ldots \ l_n\beta_{n+1}|, \ \ldots$$

and where D'_r *indicates that any one of the italic letters of* D_r *except the* r^{th} *has been replaced by the corresponding Greek letter,* D''_r *that any two letters except the* r^{th} *have been similarly treated, and so on.**

* The problem here solved was set by Nanson at the beginning of 1905 in *Educ. Times*, lviii. p. 41, and a condensed solution was published early in 1906 in *Math. from Educ. Times*, (2) ix. pp. 109–110. See also (2) x. p. 106.

There is thus brought to light an equality between two compound determinants of different orders: for example,

$$\begin{vmatrix} |\alpha_1\beta_2\gamma_3| & \Sigma|a_1\beta_2\gamma_3| & \Sigma|a_1b_2c_3| & |a_1b_2c_3| \\ |\alpha_2\beta_3\gamma_4| & \Sigma|a_2\beta_3\gamma_4| & \Sigma|a_2b_3c_4| & |a_2b_3c_4| \\ |\alpha_3\beta_4\gamma_1| & \Sigma|a_3\beta_4\gamma_1| & \Sigma|a_3b_4c_1| & |a_3b_4c_1| \\ |\alpha_4\beta_1\gamma_2| & \Sigma|a_4\beta_1\gamma_2| & \Sigma|a_4b_1c_2| & |a_4b_1c_2| \end{vmatrix}$$

$$= \begin{vmatrix} |a_1\alpha_2\beta_3\gamma_4| & |a_1\alpha_2b_3\gamma_4| + |a_1\alpha_2\beta_3c_4| & |a_1\alpha_2b_3c_4| \\ |b_1\alpha_2\beta_3\gamma_4| & |b_1\alpha_2\beta_3c_4| + |b_1\alpha_2\beta_3\gamma_4| & |b_1\alpha_2\beta_3c_4| \\ |c_1\alpha_2\beta_3\gamma_4| & |c_1\alpha_2\beta_3\gamma_4| + |c_1\alpha_2b_3\gamma_4| & |c_1\alpha_2b_3\gamma_4| \end{vmatrix}.$$

Finally, by finding the set of linear equations that are in so-called correspondence with the given set of homogeneous quadratics, the result obtained differs only from the first of the two preceding forms in being its conjugate.

SZABÓ, P. (1905/11)

[Desargues tételének analitikus bebizonyitásához. *Math. és Phys. Lapok*, xiv. pp. 316–319.]

At the outset the author takes us back to several early papers of Hunyady's on Desargues' theorem: probably all that is now necessary is the reference given by us above under Arany (1899).

MUIR, T. ($1905^{25}/_{11}$)

[Equality of two compound determinants of orders *n* and *n* —1. *Messenger of Math.*, xxxv. pp. 118–122.]

As this title is descriptive of the subject of the penultimate paragraph of the paper of $1905^6/_{11}$, the theorem of the said paragraph is here first drawn attention to. Then follows the more important cognate theorem: *If there be* n *arrays, each of* n — 1 *rows and* n *columns, and two determinants be formed therefrom, namely, one whose* rth *column has for its elements the* n *determinants of the* (n — 1)th *order derivable from the* rth *array, and the other whose* rth *column has for elements the* n — 1 *determinants of the*

nth *order formable by adding to the* rth *array one row of the* nth *array, the two determinants are equal:* for example, when n is 4,

$$\begin{vmatrix} |a_1b_2c_3| & |h_1k_2l_3| & |m_1n_2r_3| & |x_1y_2z_3| \\ |a_2b_3c_4| & |h_2k_3l_4| & |m_2n_3r_4| & |x_2y_3z_4| \\ |a_3b_4c_1| & |h_3k_4l_1| & |m_3n_4r_1| & |x_3y_4z_1| \\ |a_4b_1c_2| & |h_4k_1l_2| & |m_4n_1r_2| & |x_4y_1z_2| \end{vmatrix}$$

$$= \begin{vmatrix} |a_1b_2c_3x_4| & |h_1k_2l_3x_4| & |m_1n_2r_3x_4| \\ |a_1b_2c_3y_4| & |h_1k_2l_3y_4| & |m_1n_2r_3y_4| \\ |a_1b_2c_3z_4| & |h_1k_2l_3z_4| & |m_1n_2r_3z_4| \end{vmatrix}.$$

To connect the theorem with results already known, there are noted (1) the case where

$$h, k, l, m, n, r, x, y, z \equiv b, c, d, c, d, e, d, e, f,$$

and (2) the case where farther $e, f = a, b$.

HUNYADY, E. ($1905^1/_{12}$)

[Question 1416. *Nouv. Annales de Math.*, (4) v. pp. 568–570.]

A proof is at last here given of Hunyady's theorem of 1882 (*Hist.*, iv. p. 209). It consists in showing that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} A_1+A_1 & A_2+B_1 & A_3+C_1 \\ B_1+A_2 & B_2+B_2 & B_3+C_2 \\ C_1+A_3 & C_2+B_3 & C_3+C_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \cdot \begin{vmatrix} a_1+a_1 & a_2+b_1 & a_3+c_1 \\ b_1+a_2 & b_2+b_2 & b_3+c_2 \\ c_1+a_3 & c_2+b_3 & c_3+c_3 \end{vmatrix}$$

and this is accomplished by performing the two multiplications indicated, the one rowwise and the other columnwise. The name of the ingenious author is not given.

NANSON, E. J. (1906 28/3)

[A theorem in compound determinants. *Messenger of Math.*, xxxvi. pp. 45–48.]

This follows immediately on Muir's second paper of the previous year, its main object being to show how the results of that paper may be deduced from a single general equality, namely, the equality now known as the unextended case of Bazin's theorem of 1851. Whether the new formulæ so founded are equally general with those sought to be reproduced it is a little difficult to say, because of uncertainty as to the meaning of the new notations employed; for example, whether

$$\begin{vmatrix} (a_4b_1c_1d_1) & (b_4b_1c_1d_1) & (c_4b_1c_1d_1) \\ (a_2a_4c_2d_2) & (a_2b_4c_2d_2) & (a_2c_4c_2d_2) \\ (a_3b_3a_4d_3) & (a_3b_3b_4d_3) & (a_3b_3c_4d_3) \end{vmatrix} = (A_1B_2C_3D_4)$$

is coextensive with

$$\begin{vmatrix} |a_1b_2c_3x_4| & |h_1k_2l_3x_4| & |m_1n_2r_3x_4| \\ |a_1b_2c_3y_4| & |h_1k_2l_3y_4| & |m_1n_2r_3y_4| \\ |a_1b_2c_3z_4| & |h_1k_2l_3z_4| & |m_1n_2r_3z_4| \end{vmatrix}$$

$$= \begin{vmatrix} |a_1b_2c_3| & |h_1k_2l_3| & |m_1n_2r_3| & |x_1y_2z_3| \\ |a_2b_3c_4| & |h_2k_3l_4| & |m_2n_3r_4| & |x_2y_3z_4| \\ |a_3b_4c_1| & |h_3k_4l_1| & |m_3n_4r_1| & |x_3y_4z_1| \\ |a_4b_1c_2| & |h_4k_1l_2| & |m_4n_1r_2| & |x_4y_1z_2| \end{vmatrix}.$$

The interesting derivation of Sylvester's theorem regarding an aggregate of products of pairs of determinants we have already seen handled by Rubini in 1878 (*Hist.*, iii. p. 200).

PETR, K. (1906)

[Několik poznámek o determinantech. *Časopis pro pěstování math. a fys.*, xxxv. pp. 311–321: or, in Magyar, *Math. és Phys. Lapok*, xv. pp. 353–365: or, in German, *Math.-naturw. Berichte aus Ungarn*, xxv. pp. 95–105.]

The second section (pp. 97–103) of this paper concerns the latent roots of the compounds of a determinant. First, there is established Rados' theorem of 1891 (*Hist.*, iv. pp. 215, 217), the procedure being based on the theory of linear substitution. In the same manner and at almost equal length a wider theorem is then successfully dealt with, namely: *If* $|\mathrm{a}_{hk}\mathrm{x} + \mathrm{b}_{hk}\mathrm{y}|_n$ *be resolvable into* n *linear factors of the form* $\alpha\mathrm{x} + \beta\mathrm{y}$, *and* $\mathrm{a}_{pq}^{(m)}$ *be any element of the* m*th* *compound of* $|\mathrm{a}_{hk}|_n$, *then* $|\mathrm{a}_{pq}^{(m)}\mathrm{x} + \mathrm{b}_{pq}^{(m)}\mathrm{y}|_{(n)_m}$ *is similarly resolvable into* $(\mathrm{n})_m$ *factors.* A purely determinantal proof of the latter is a desideratum.

JAHNKE, E. (1906⁸/₁₁)

[Aufgaben und Lehrsätze. Nr. 125. *L'Intermédiaire des Math.*, xii. p. 172: *Archiv d. Math. u. Phys.* (3) ix. p. 91; (3) xi. p. 140.]

The subject here is a quadric relation between six 5-line minors of the second compound of $|a_1b_2c_3d_4|$. Grassmann's semi-geometrical method is used, but the relation is easily established otherwise. Calling the compound in question S, we have from Franke's theorem

$$\text{cof } |a_1b_2| \text{ in } S = |a_1b_2c_3d_4|^2 \cdot |c_3d_4|,$$

and thus

$$\begin{aligned}\text{cof}\,|a_1b_2|\cdot\text{cof}\,|a_3b_4| - \text{cof}\,|a_1b_3|\cdot\text{cof}\,|a_2b_4| + \text{cof}\,|a_1b_4|\cdot\text{cof}\,|a_2b_3| \\ = |a_1b_2c_3d_4|^4\cdot\{-|c_1d_2|\,|c_3d_4| + |c_1d_3|\,|c_2d_4| - |c_1d_4|\,|c_2d_3|\} \\ = |a_1b_2c_3d_4|^4\cdot 0 = 0.\end{aligned}$$

It will be noted that the six 5-line minors of S cannot be taken arbitrarily, but are those which are cofactors of elements of a single column of S.

MUIR, T. (1906¹²/₁₁)

[The minors of a product determinant. *Proceed. R. Soc. Edinburgh*, xxvii. pp. 79–87.]

Incidentally light is here (§ 6) thrown on Hesse's generalization of the equality

$$\begin{vmatrix} u_{11} & u_{12} & \alpha_1 \\ u_{21} & u_{22} & \alpha_2 \\ \alpha_1 & \alpha_2 & \cdot \end{vmatrix} \begin{vmatrix} u_{11} & u_{12} & \gamma_1 \\ u_{21} & u_{22} & \gamma_2 \\ \gamma_1 & \gamma_2 & \cdot \end{vmatrix} - \begin{vmatrix} u_{11} & u_{12} & \alpha_1 \\ u_{21} & u_{22} & \alpha_2 \\ \gamma_1 & \gamma_2 & \cdot \end{vmatrix}^2 = \left| u_{11}\, u_{22} \right| \cdot \left| \alpha_1\, \gamma_2 \right|^2,$$

a theorem which at one time attracted considerable attention but which was left without the cofactor of $|u_{11}\, u_{22} \ldots u_{nn}|$ on the right being determined (*Hist.*, i. pp. 129–132). Taking the single determinant equal to

$$| a_1b_2c_3d_4 | \cdot | F_1G_2H_3K_4 | \cdot | m_1n_2r_3s_4 |$$

where $| F_1G_2H_3K_4 |$ is the adjugate of $|f_1g_2h_3k_4|$, the author in the first place shows that a 2-line minor of the product can be expressed with elements that are bordered determinants, and then by using the main theorem of his paper he arrives at the result

$$\left\| \begin{matrix} \begin{vmatrix} \cdot & a_1 & a_2 & a_3 & a_4 \\ m_2 & f_1 & g_1 & h_1 & k_1 \\ n_2 & f_2 & g_2 & h_2 & k_2 \\ r_2 & f_3 & g_3 & h_3 & k_3 \\ s_2 & f_4 & g_4 & h_4 & k_4 \end{vmatrix} & \begin{vmatrix} \cdot & a_1 & a_2 & a_3 & a_4 \\ m_3 & f_1 & g_1 & h_1 & k_1 \\ n_3 & f_2 & g_2 & h_2 & k_2 \\ r_3 & f_3 & g_3 & h_3 & k_3 \\ s_3 & f_4 & g_4 & h_4 & k_4 \end{vmatrix} \\ \begin{vmatrix} \cdot & c_1 & c_2 & c_3 & c_4 \\ m_2 & f_1 & g_1 & h_1 & k_1 \\ n_2 & f_2 & g_2 & h_2 & k_2 \\ r_2 & f_3 & g_3 & h_3 & k_3 \\ s_2 & f_4 & g_4 & h_4 & k_4 \end{vmatrix} & \begin{vmatrix} \cdot & c_1 & c_2 & c_3 & c_4 \\ m_3 & f_1 & g_1 & h_1 & k_1 \\ n_3 & f_2 & g_2 & h_2 & k_2 \\ r_3 & f_3 & g_3 & h_3 & k_3 \\ s_3 & f_4 & g_4 & h_4 & k_4 \end{vmatrix} \end{matrix} \right\| = | f_1g_2h_3k_4 | \, B_6$$

where

$$B_6 = \begin{array}{cccc|c} |a_1c_2| & |a_1c_3| & \ldots & |a_3c_4| & \\ \hline |h_3k_4| & -|g_3k_4| & \ldots & |f_3g_4| & |m_2n_3| \\ -|h_2k_4| & |g_2k_4| & \ldots & -|f_2g_4| & |m_2r_3| \\ \cdot & \cdot & \cdot & \cdot & \\ |h_1k_2| & -|g_1k_2| & \ldots & |f_1g_2| & |r_2s_3| \end{array}.$$

The compound determinant on the left degenerates into Hesse's when it and $|f_1g_2h_3k_4|$ are made axisymmetric; and the bilinear or bipartite form, B_6, becomes what Hesse would have obtained had he pursued his work.

MUIR, T. ($1907^{20}/_8$)

[The theory of compound determinants in the historical order of its development up to 1860. *Proceed. R. Soc. Edinburgh*, xxviii. pp. 197–209.]

This, our first historical paper on the present subject, is introduced by a few paragraphs pointing out the early foreshadowings of the "adjugate" by Lagrange (1773), the first direct investigations of the same by Cauchy and by Jacobi, Cauchy's advance to other "systèmes dérivés" of $|a_{1n}|$ besides the adjugate, and Jacobi's carrying of the idea into the special field of "functional determinants". Then follow separate notices of three of Sylvester's papers and the text-books of Spottiswoode, Brioschi, and Bellavitis.

MUIR, T. ($1907^{7}/_{10}$)

[The product of the primary minors of an m-by-$(m+1)$ array. *Proceed. R. Soc. Edinburgh*, xxviii. pp. 210–216.]

The second part of this paper concerns compound determinants of a special type, namely, those whose elements are products of pairs of determinants. The first to be considered is of order $\frac{1}{2}m(m-1)$ and whose component determinants are 2-line minors of an m-by-$(m+1)$ array; for example, when m is 4 and the array is

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5, \end{matrix}$$

the equality is

$$\begin{vmatrix} |a_1b_2|\cdot|a_1b_3| & |a_1c_2|\cdot|a_1c_3| & \dots & |c_1d_2|\cdot|c_1d_3| \\ |a_1b_2|\cdot|a_1b_4| & |a_1c_2|\cdot|a_1c_4| & \dots & |c_1d_2|\cdot|c_1d_4| \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ |a_1b_4|\cdot|a_1b_5| & |a_1c_4|\cdot|a_1c_5| & \dots & |c_1d_4|\cdot|c_1d_5| \end{vmatrix}$$
$$= a_1{}^2b_1{}^2c_1{}^2d_1{}^2\cdot|a_1b_2c_3d_4|\cdot|a_1b_2c_3d_5|\cdot|a_1b_2c_4d_5|\cdot|a_1b_3c_4d_5|.$$

The prominence of a_1, b_1, c_1, d_1 on the right is due to the fact that the component 2-line minors on the left all have their first columns

coincident with the first column of the array. No direct proof is given.

Next comes a series of results in which the component determinants are the primary minors of $|a_1b_2c_3d_4|$, or Δ say. Among the more interesting are

$$\begin{vmatrix} A_1A_2 & A_2A_3 & A_3A_4 & A_4A_1 \\ B_1B_2 & B_2B_3 & B_3B_4 & B_4B_1 \\ C_1C_2 & C_2C_3 & C_3C_4 & C_4C_1 \\ D_1D_2 & D_2D_3 & D_3D_4 & D_4D_1 \end{vmatrix} = \Delta^2 \begin{vmatrix} a_2A_2 & a_4A_2 & a_4A_4 & a_2A_4 \\ b_2B_2 & b_4B_2 & b_4B_4 & b_2B_4 \\ c_2C_2 & c_4C_2 & c_4C_4 & c_2C_4 \\ d_2D_2 & d_4D_2 & d_4D_4 & d_2D_4 \end{vmatrix}$$
$$= -\Delta^4 \,|\, a_1a_2 \quad b_2b_3 \quad c_3c_4 \quad d_4d_1 \,|;$$

and

$$|\, A_2A_3A_4 \quad B_3B_4B_1 \quad C_4C_1C_2 \quad D_1D_2D_3 \,|$$
$$= \Delta^2 \,|\, a_2A_3A_4 \quad b_1B_3B_4 \quad C_1C_2c_4 \quad D_1D_2d_3 \,|.$$

MUIRHEAD, R. F. ($1908^{13}/_3$)

[To express a determinant of the n^{th} order in terms of compound determinants of the 2^{nd} order. *Proceed. Edinburgh Math. Soc.*, xxvi. pp. 15–17.]

The series of 2-line determinants here,

$$T_1, T_2, T_3, \ldots,$$

is such that the elements of T_2 are quantities like T_1, the elements of T_3 are quantities like T_2, and so on: for example,

$$a_1, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \begin{vmatrix} |a_1b_2| & |a_1b_3| \\ |a_1c_2| & |a_1c_3| \end{vmatrix}, \ldots$$

where T_1 is necessarily an alien, and $T_3, T_4, T_5. \ldots$ notwithstanding their increasing complexity, are quite simply shown to be equal to

$$a_1\,|a_1b_2c_3|, \quad a_1^2\,|a_1b_2|\cdot|a_1b_2c_3d_4|,$$
$$a_1^4\,|a_1b_2|^2\cdot|a_1b_2c_3|\cdot|a_1b_2c_3d_4e_5|, \quad \ldots$$

The result thus obtained is

$$|a_1b_2c_3d_4e_5| = T_5^{\,1}T_4^{\,0}T_3^{\,-1}T_2^{\,-2}T_1^{\,-3},$$

and so generally.

FISCHER, E. ($1909^{15}/_2$)

[Verallgemeinerung des Sylvesterschen Determinantensatzes. *Crelle's Journ.*, cxxxv. pp. 306–318.]

The title of this interesting paper is not helpful as to its contents, the actual subject being the problem of the factorization of minors of the m^{th} compound of $|a_{1n}|$. Further, the problem is restricted in two directions: the minors of the compound have to be coaxial and their factors have to be leading coaxial minors of $|a_{1n}|$. For illustrative examples it is perhaps most convenient to go to the exposition of 1853 by Spottiswoode (*Hist.*, ii. p. 203), whom, by the way, Fischer might more appropriately have referred to than to Sylvester. When n is 5 and m is 3, and to construct our minor we take all the triads formable from the first five integers except 245, 345, we obtain from Spottiswoode's first equality

$$\left| \begin{vmatrix}123\\123\end{vmatrix} \begin{vmatrix}124\\124\end{vmatrix} \begin{vmatrix}125\\125\end{vmatrix} \begin{vmatrix}134\\134\end{vmatrix} \begin{vmatrix}135\\135\end{vmatrix} \begin{vmatrix}145\\145\end{vmatrix} \begin{vmatrix}234\\234\end{vmatrix} \begin{vmatrix}235\\235\end{vmatrix} \right|$$
$$= \begin{vmatrix}1\\1\end{vmatrix} \cdot \begin{vmatrix}123\\123\end{vmatrix} \cdot \begin{vmatrix}12345\\12345\end{vmatrix}^4,$$

and leaving out the two additional triads 145, 235, we obtain from his fourth equality

$$\left| \begin{vmatrix}123\\123\end{vmatrix} \begin{vmatrix}124\\124\end{vmatrix} \begin{vmatrix}125\\125\end{vmatrix} \begin{vmatrix}134\\134\end{vmatrix} \begin{vmatrix}135\\135\end{vmatrix} \begin{vmatrix}234\\234\end{vmatrix} \begin{vmatrix}235\\235\end{vmatrix} \right|$$
$$= \begin{vmatrix}123\\123\end{vmatrix}^2 \cdot \begin{vmatrix}12345\\12345\end{vmatrix}^3.$$

Sets of combinations, such as we have here, namely,

123, 124, 125, 134, 135, 145, 234, 235,

and

123, 124, 125, 134, 135, 234, 235,

are said to be "closed" or "compact" sets, because the replacing of any integer in any combination of the set by a lower integer does not produce a combination which is not already there.

With this explanation the reader will readily understand our

formulation of the theorem to which the paper under consideration is devoted. *The necessary and sufficient condition that a given coaxial minor of the* mth *compound of an* n-*line determinant be expressible as a product of powers of leading coaxial minors of the said determinant is that the set of* m-*ads of* 1, 2, 3, . . . , n *involved in the specification of the given minor be a "closed" set.* The proof (pp. 309–313) is based on work of Mertens', dating from 1880, on the conditions for the divisibility of any rational integral function of the elements of $|a_{1n}|$ by a power of $|a_{1n}|$ (*Hist.*, iv. p. 31).

Nothing is said about the finding of the exponents of the powers in the result of the factorization, nor indeed is a single example vouchsafed of the application of the theorem. This is the more to be regretted because a dangerously plausible rule for obtaining them lies suggestively on the surface. For example, the given closed set of combinations being

$$123,\quad 124,\quad 125,\quad 134,\quad 135,\quad 145,\quad 234,$$

and it being noted that the integers of the set may be combined thus, $1^6 2^4 3^4 4^4 5^3$, and redistributed thus, $(12345)^3\,(1234)\,(1)^2$, we feel inclined to conclude that

$$\left|\;\begin{vmatrix}123\\123\end{vmatrix}\;\begin{vmatrix}124\\124\end{vmatrix}\;\begin{vmatrix}125\\125\end{vmatrix}\;\begin{vmatrix}134\\134\end{vmatrix}\;\begin{vmatrix}135\\135\end{vmatrix}\;\begin{vmatrix}145\\145\end{vmatrix}\;\begin{vmatrix}234\\234\end{vmatrix}\;\right|$$
$$= \begin{vmatrix}1\\1\end{vmatrix}^2 \cdot \begin{vmatrix}1234\\1234\end{vmatrix} \cdot \begin{vmatrix}12345\\12345\end{vmatrix}^3;$$

and such indeed is the case. But the same procedure being followed in the case of the unclosed set

$$123,\quad 124,\quad 134,\quad 135,\quad 145,\quad 234,\quad 235,\quad 245,$$

we should be wrong to conclude, as Spottiswoode actually does, that the result is

$$\begin{vmatrix}1234\\1234\end{vmatrix}\;\begin{vmatrix}12345\\12345\end{vmatrix}^4,$$

Franke's theorem of 1862 affording a check by giving the true cofactor of $\begin{vmatrix}12345\\12345\end{vmatrix}^4$.

The remaining pages of the paper are occupied in showing that a quite similar theorem holds in the case of the Reiss-Picquet "mixed" compound (*Hist.*, iii. pp. 198–199).

KOWALEWSKI, G. (1909/$_4$)

[Einführung in die Determinantentheorie . . . vi + 550 pp. Leipzig.]

Half of the sixteen pages (pp. 83–88, 99–111) devoted to the consideration of the adjugate and other compound determinants is taken up, strangely enough, with three independent proofs of Sylvester's ungeneralized theorem of 1851 (*Hist.*, ii. pp. 59–61, 193–194). First, it is viewed as a condensation-theorem, the procedure being that with which we are already familiar (*Hist.*, iii. p. 81). In the second proof the fundamental step is the formation of a bordered determinant of which the compound concerned is both a minor and a factor: for example, to prove the simple extensional

$$\Big|\; |a_1b_2c_3d_4| \quad |a_1b_2c_3e_5| \quad |a_1b_2c_3f_6| \quad |a_1b_2c_3g_7| \;\Big| = |a_1b_2c_3|^3 \cdot |a_1b_2c_3d_4e_5f_6g_7|,$$

a determinant equal to

$$|a_1b_2c_3| \cdot \Big|\; |a_1b_2c_3d_4| \quad |a_1b_2c_3e_5| \quad |a_1b_2c_3f_6| \quad |a_1b_2c_3g_7| \;\Big|$$

would be formed, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \cdot & \cdot & \cdot & |a_1b_2c_3d_4| & |a_1b_2c_3d_5| & |a_1b_2c_3d_6| & |a_1b_2c_3d_7| \\ \cdot & \cdot & \cdot & |a_1b_2c_3e_4| & |a_1b_2c_3e_5| & |a_1b_2c_3e_6| & |a_1b_2c_3e_7| \\ \cdot & \cdot & \cdot & |a_1b_2c_3f_4| & |a_1b_2c_3f_5| & |a_1b_2c_3f_6| & |a_1b_2c_3f_7| \\ \cdot & \cdot & \cdot & |a_1b_2c_3g_4| & |a_1b_2c_3g_5| & |a_1b_2c_3g_6| & |a_1b_2c_3g_7| \end{vmatrix}$$

Then row_4 of this determinant would be increased by

$$\text{row}_1 \cdot |b_1c_2d_3| - \text{row}_2 \cdot |a_1c_2d_3| + \text{row}_3 \cdot |a_1b_2d_3|,$$

and would thus take the form

$$d_1\,|a_1b_2c_3|, \quad d_2\,|a_1b_2c_3|, \quad d_3\,|a_1b_2c_3|, \quad \ldots, \quad d_7\,|a_1b_2c_3|.$$

Similarly row_5 would be treated, becoming

$$e_1\,|a_1b_2c_3|, \quad e_2\,|a_1b_2c_3|, \quad e_3\,|a_1b_2c_3|, \quad \ldots, \quad e_7\,|a_1b_2c_3|,$$

and so with the others. The determinant would then be seen to be equal to

$$| a_1b_2c_3 |^4 \cdot | a_1b_2c_3d_4e_5f_6g_7 |;$$

and nothing further would be necessary save division of both of its equivalents by $| a_1b_2c_3 |$. The third proof is still less direct, arising from the equatement of two expressions obtainable for the adjugate of a minor of the adjugate of any determinant: for example, the basic determinant being, as in the preceding, $| a_1b_2c_3d_4e_5f_6g_7 |$, or Δ say, two expressions can be obtained for the adjugate of $| D_4E_5F_6G_7 |$, namely,

$$| D_4E_5F_6G_7 |^3 \quad \text{i.e.} \quad \left\{ | a_1b_2c_3 | \cdot \Delta^3 \right\}^3 \quad \text{i.e.} \quad | a_1b_2c_3 |^3 \cdot \Delta^9,$$

and, by removing Δ^2 from each element,

$$(\Delta^2)^4 \Big| \; | a_1b_2c_3d_4 | \quad | a_1b_2c_3e_5 | \quad | a_1b_2c_3f_6 | \quad | a_1b_2c_3g_7 | \; \Big|,$$

the equating of which two expressions shows that the compound determinant is equal to

$$| a_1b_2c_3 |^3 \cdot \Delta$$

as required. This third mode of proof is again used to establish the generalized theorem (*Hist.*, ii. pp. 194–197): in essentials it does not differ from Baker's mode of 1904 (see above, pp. 218–9). As for the two other theorems dealt with, we only note that the names given to them deserve reconsideration.

MUIR, T. (1912 $^{20}/_{3}$)

[The resultant of a set of homogeneous lineo-linear equations. *Transac. R. Soc. S. Africa*, ii. pp. 373–380.]

This seeks, for one thing, to clear up the haze surrounding Sylvester's doings with "double determinants", entities first spoken of by him with heated enthusiasm and then quietly dropped (*Hist.*, iii. pp. 178–179). At the outset are obtained the two forms of resultant which he had spoken of as being unsatisfactory, and then a justifiable guess is made as to the unpublished form which more than met with his approval. A simple mode of obtaining this form is given, and the process is shown to be of

perfectly general application, the set of equations considered with a view to the elimination of the $m+n$ unknowns being

$$0 = \left(\begin{array}{cccc} a_1 & a_2 & \ldots & a_{mn} \\ b_1 & b_2 & \ldots & b_{mn} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \multicolumn{4}{c}{(m+n-1 \text{ rows})} \end{array}\right\rangle x_1, x_2, \ldots, x_m)\,(y_1, y_2, \ldots, y_n)$$

where

$$(\alpha, \beta, \gamma, \ldots)\,(p, q, r, \ldots)$$

is used to stand for

$$\alpha p, \beta p, \gamma p, \ldots, \alpha q, \beta q, \gamma q, \ldots, \alpha r, \beta r, \gamma r, \ldots$$

Special attention is given to the case where $m, n = 3, 3$.

VIVANTI, G. ($1912^{23}/_5$): SIBIRANI, F. ($1913^3/_7$)

[Un teorema sui determinanti. *Rendic. . . . Ist. Lombardo*, . . . (2) xlv. pp. 556–559.]

[Un teorema sui determinanti. *Rendic. . . . Ist. Lombardo*, . . . (2) xlvi. pp. 822–830.]

The determinant here dealt with is of the order $\frac{1}{2}m(m+1)$, every row of it consisting of the squares and other binary products of m magnitudes. Simpler instances of it we have seen appearing in the work of Hunyady and others: for example,

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & y_1z_1 & z_1x_1 & x_1y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2z_2 & z_2x_2 & x_2y_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_6^2 & y_6^2 & z_6^2 & y_6z_6 & z_6x_6 & x_6y_6 \end{vmatrix}$$

(*Hist.*, iii. pp. 202–204). The important difference now is that instead of the simple variables $x_1, y_1, \ldots$ Vivanti has determinants, namely, the $(m-1)$-line minors of an m-by-$(m+1)$ array, his theorem being a generalization of the equality

$$\begin{vmatrix} a_1{}^2 & b_1{}^2 & a_1b_1 \\ a_2{}^2 & b_2{}^2 & a_2b_2 \\ a_3{}^2 & b_3{}^2 & a_3b_3 \end{vmatrix} = |a_1b_2| \cdot |a_2b_3| \cdot |a_3b_1|$$

(*Hist.*, iii. p. 132), and expressible as follows: *The determinant of order* $\frac{1}{2}$m(m + 1) *whose every row is of the form*

$$a^2 \quad b^2 \quad c^2 \quad \ldots \quad y^2 \quad z^2 \quad ab \quad ac \quad \ldots \quad yz,$$

and whose first m *columns have for elements the squares of the* (m — 1)*-line minors of an* m*-by-*(m + 1) *array is equal to the* (m — 1)[th] *power of the product of the* m*-line minors of the array*; for example, when m is 3 and the array is

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix}$$

we have

$$\begin{vmatrix} |a_1b_2|^2 & |a_1c_2|^2 & |b_1c_2|^2 & |a_1b_2|\cdot|a_1c_2| & |a_1b_2|\cdot|b_1c_2| & |a_1c_2|\cdot|b_1c_2| \\ |a_1b_3|^2 & |a_1c_3|^2 & |b_1c_3|^2 & |a_1b_3|\cdot|a_1c_3| & |a_1b_3|\cdot|b_1c_3| & |a_1c_3|\cdot|b_1c_3| \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ |a_3b_4|^2 & |a_3c_4|^2 & |b_3c_4|^2 & |a_3b_4|\cdot|a_3c_4| & |a_3b_4|\cdot|b_3c_4| & |a_3c_4|\cdot|b_3c_4| \end{vmatrix}$$

$$= |a_1b_2c_3|^2 \cdot |a_1b_2c_4|^2 \cdot |a_1b_3c_4|^2 \cdot |a_2b_3c_4|^2.$$

The proof consists in showing, from an examination of the 1[st], 2[nd], 4[th] rows, that $|a_1b_2c_3|^2$ is a factor, then similarly that $|a_1b_2c_4|^2$, $|a_1b_3c_4|^2$, $|a_2b_3c_4|^2$ are factors, and finally that the only other possible factor is a power of —1.

As bringing additional interest to an already interesting theorem we draw attention to the fact that the product of the m-line minors of an m-by-$(m + 1)$ array was shown in 1907 to be itself expressible as a determinant of order $\frac{1}{2}m(m + 1)$, and that consequently we have the curious result that one special determinant of that order equals the $(m — 1)$[th] power of another: for example, when m is 3 the 6-line compound determinant above

$$= \begin{vmatrix} a_1a_2 & b_1b_2 & c_1c_2 & a_1b_2 + a_2b_1 & a_1c_2 + a_2c_1 & b_1c_2 + b_2c_1 \\ a_1a_3 & b_1b_3 & c_1c_3 & a_1b_3 + a_3b_1 & a_1c_3 + a_3c_1 & b_1c_3 + b_3c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_3a_4 & b_3b_4 & c_3c_4 & a_3b_4 + a_4b_3 & a_3c_4 + a_4c_3 & b_3c_4 + b_4c_3 \end{vmatrix}^2.$$

An independent proof of this would be of value.

Sibirani's paper is avowedly a close following up of Vivanti's, the key to the generalization made lying in the fact that the originating array is no longer of $m+1$ columns but of $m+p$. The order of the compound determinant is now $(m+p)_{p+1}$: the minors that go to the construction of it are as before of the order $m-1$, but the number of them involved in an element is no longer 2 but $p+1$. The right-hand side of the equality is in keeping with this; the minors involved are as before of the order m and the index of the power to which they are raised is $m-1$, but the number of them is now $(m+p)_m$.

METZLER, W. H. ($1913^{17}/_4$)

[Some factorable minors of a compound determinant. *Proceed. R. Soc. Edinburgh*, xxxiv. pp. 27–31.]

The avowed object of this paper is to include in one generalization Sylvester's theorem of 1851 (*Hist.*, ii. pp. 193–197) and a theorem of Muir's. Unfortunately, the width of the generalization makes the statement of it distinctly troublesome: it and the explanations of its attendant symbolisms occupy a full page and a half (pp. 29–31) of the paper. We give instead an example where $m=5$ and $k=3$, the minor proposed for factorization being

$$\Big|\ |a_xb_yc_z|\ \ |a_xb_yd_4|\ \ |a_xb_ye_5|\ \ |a_xc_3d_4|\ \ |a_xc_3e_5|\ \ |a_xd_4e_5|\ \ |b_2c_3d_4|\ \ |b_2c_3e_5|\ \ |b_2d_4e_5|\ \ |c_3d_4e_5|\ \Big|,$$

or M say.

Multiplying M by $|a_1b_2c_3d_4e_5|^4$ in the form of the 2^{nd} compound of $|a_1b_2c_3d_4e_5|$, namely,

$$\Big|\ |d_4e_5|\ \ |c_3e_5|\ \ |c_3d_4|\ \ |b_2e_5|\ \ |b_2d_4|\ \ |b_2c_3|\ \ |a_1e_5|\ \ |a_1d_4|\ \ |a_1c_3|\ \ |a_1b_2|\ \Big|$$

with its 2^{nd}, 5^{th}, 7^{th}, 9^{th} rows altered in sign, we obtain a product determinant having nothing but zeros on one side of the diagonal, and its diagonal term equal to

$$|a_xb_yc_zd_4e_5|\cdot|a_xb_yc_4d_3e_5|\cdot|a_xb_yc_5d_3e_4|\cdot|a_xb_3c_4d_2e_5|\cdot|a_xb_3c_5d_2e_4|\cdot|a_xb_4c_5d_2e_3|\cdot|a_1b_2c_3d_4e_5|^4.$$

The result thus reached is *

$$M = \left| a_x b_y c_z d_4 e_5 \right| \cdot \left| a_x b_y c_3 d_4 e_5 \right|^2 \cdot \left| a_x b_2 c_3 d_4 e_5 \right|^3.$$

METZLER, W. H. (1914¹/₃)

[On the expansion of certain minors of the L^{th} compound of a determinant as a function of the elements of a single line of the m^{th} compound. *Annals of Math.*, xv. pp. 166–169.]

This paper follows up Muir's paper of 1905 on the equality of two compound determinants. The nature of the generalization effected will be understood from the fact that it depends on the multiplication of the k^{th} compound by a minor of the $(m-k)^{\text{th}}$ compound without the resulting determinant being a minor of the m^{th} compound. Specializations leading to factorization of the product-determinant afford results of varying interest. The example given is

$$\Big| \; |a_1b_2| \;\; |a_3c_4| \;\; |a_5d_6| \;\; |b_7c_8| \;\; |b_9d_\tau| \;\; |c_\zeta d_\sigma| \; \Big| \cdot |a_9 b_\tau c_\sigma d_\zeta|$$
$$= \Big| \; |a_1b_2c_\tau d_\zeta| \;\; |a_3b_4c_\tau d_\sigma| \;\; |a_5b_6c_9d_\zeta| \;\; |a_7b_8c_9d_\sigma| \; \Big|.$$

It may be readily verified by multiplying the 6-line determinant on the left by the second compound of $|a_9 b_\zeta c_\sigma d_\tau|$ and striking out $|a_9 b_\zeta c_\sigma d_\tau|^2$ from both sides.

The other theorem of 1905 is also generalized.†

THAER, F. (1914): THAER, C. (1914)

[Ein Determinantensatz. *Mittheil. d. math. Ges. zu Hamburg*, v. pp. 141–142.]

[Ein Determinantensatz. *Archiv d. Math. u. Phys.*, (3) xxiii. pp. 92–93.]

The theorem in question asserts the divisibility of

* To be strictly accurate it should be added that Metzler's equality is the complementary of this, and therefore has an additional factor on the right-hand side. His notation for minors is unusual, $\left|\begin{smallmatrix}1&2&3\\ m&n&p\end{smallmatrix}\right|$ standing for the complementary of what is ordinarily denoted by that symbol—standing, in fact, for our $\vdots\begin{smallmatrix}1&2&3\\ m&n&p\end{smallmatrix}\vdots$ (*Transac. R. Soc. Edinburgh*, xl. pp. 511–533).

† These wide generalizations of Metzler's might well have been referred to at the end of a recent paper in the *Proceedings R.S.E.* (xlv. pp. 51–55): the effect of the footnote there given would thus have been much enhanced.

$|A^p_{11}A^p_{22}\ldots A^p_{nn}|$ by $|a_{11}a_{22}\ldots a_{nn}|^{n-1}$. Probably a better proof than those here given is readily got by using Chio's condensation-theorem (*Hist.*, ii. pp. 79–81).

MUIR, T. ($1914^{16}/_9$)

[Note on Hesse's generalization of Pascal's theorem. *Transac. R. Soc. S. Africa*, (2) v. pp. 39–43.]

This concerns Hesse's equalities of 1872 (*Hist.*, iii. pp. 437–438) regarding the determinants $[\alpha, \beta]$, $[\alpha\gamma, \beta\delta]$ got by the bordering of $|u_{1n}|$. It is first shown that Hesse's result

$$|u_{1n}|\cdot[\alpha\gamma, \beta\delta] = [\alpha\beta, \gamma\delta] - [\alpha\delta, \gamma\beta]$$

can be followed up by

$$|u_{1n}|\cdot[\alpha\gamma\epsilon, \beta\delta\zeta] = [\alpha, \beta]\cdot[\gamma\epsilon, \delta\zeta] + [\alpha, \delta]\cdot[\gamma\epsilon, \beta\zeta] + [\alpha, \zeta]\cdot[\gamma\epsilon, \beta\delta],$$

and others like it: then that the same result written in the form

$$|u_{1n}|\cdot[\alpha\gamma, \beta\delta] = \begin{vmatrix} [\alpha, \beta] & [\alpha, \delta] \\ [\gamma, \beta] & [\gamma, \delta] \end{vmatrix}$$

is also the opening member of another series, of which the next member is

$$|u_{1n}|^2\cdot[\alpha\gamma\epsilon, \beta\delta\zeta] = \begin{vmatrix} [\alpha, \beta] & [\alpha, \delta] & [\alpha, \zeta] \\ [\gamma, \beta] & [\gamma, \delta] & [\gamma, \zeta] \\ [\epsilon, \beta] & [\epsilon, \delta] & [\epsilon, \zeta] \end{vmatrix}:$$

and finally that this second series can by despecialization be led up to an already well-known " extensional " (*Hist.*, iv. p. 8).

PASCAL, M. ($1914/_{11}$)

[Un teorema relativo ai determinanti composti. *Giornale di Mat.*, lii. pp. 372–376.]

The first three pages of this paper are occupied with the two series of so-called " condensation-theorems " already more than once rediscovered (see above, p. 220). In the remaining two pages, however, an interesting deduction, not hitherto brought to light, is drawn attention to and discussed. It is arrived at quite simply

by suitably permuting the columns of the basic determinant. For example, by Hermite's initial theorem of one of the said series, we have

$$|a_1b_2c_3d_4| \cdot a_2a_3 = \begin{vmatrix} |a_1b_2| & |a_2b_3| & |a_3b_4| \\ |a_1c_2| & |a_2c_3| & |a_3c_4| \\ |a_1d_2| & |a_2d_3| & |a_3d_4| \end{vmatrix},$$

and from this and the companion equality got by mere interchange of the suffixes 1 and 4 we deduce

$$\Big| \; |a_1b_2| \quad |a_2c_3| \quad |a_3d_4| \; \Big| = \Big| \; |a_2b_4| \quad |a_2c_3| \quad |a_3d_1| \; \Big|.$$

The question of proving such an equality otherwise is not entered on.

MUIR, T. (1915¹²/₁₁)

(See under this heading in Chapter I.)

MUIR, T. (1916¹/₅, 1917¹/₆, 1917¹/₇)

[Question 18215. *Math. Quest. and Sol.*, ii. pp. 35–36.]

[Question 18448. *Math. Quest. and Sol.*, iii. part 6.]

[Question 18467. *Math. Quest. and Sol.*, iv. pp. 75–76.]

Of the three results here the first and second are the more interesting, the third being merely an alternative proof of a known equality. The first effects the resolution of

$$\begin{vmatrix} |x_1y_\alpha| & |x_2y_\alpha| & |x_1y_\beta| & |x_2y_\beta| & |x_1y_\gamma| & |x_2y_\gamma| \\ |x_3y_\alpha| & |x_4y_\alpha| & |x_3y_\beta| & |x_4y_\beta| & |x_3y_\gamma| & |x_4y_\gamma| \\ |x_5y_\alpha| & |x_6y_\alpha| & |x_5y_\beta| & |x_6y_\beta| & |x_5y_\gamma| & |x_6y_\gamma| \end{vmatrix}$$

into

$$|x_\alpha y_\beta| \cdot |x_\beta y_\gamma| \cdot |x_\gamma y_\alpha| \cdot \begin{vmatrix} x_1x_2 & y_1y_2 & x_1y_2+x_2y_1 \\ x_3x_4 & y_3y_4 & x_3y_4+x_4y_3 \\ x_5x_6 & y_5y_6 & x_5y_6+x_6y_5 \end{vmatrix};$$

and the second is the theorem that in the second compound of any 4-line determinant we may have one 3-line minor equal to

another although differing from it in form and situation, namely:

$$\begin{vmatrix} p & 7-p & r \\ p & 7-p & r \end{vmatrix} = \begin{vmatrix} q & 7-q & r \\ q & 7-q & r \end{vmatrix}$$

where p, $7-p$, q, $7-q$, r are the ordinal numbers of rows and columns which the minors occupy in the compound.

WHITTAKER, E. T. ($1916^1/_{11}$)

(See under this heading in Chapter on Latent Roots.)

MUIR, T. ($1917^{11}/_4$)

[Note on the resolvability of the minors of a compound determinant. *Transac. R. Soc. S. Africa*, vii. pp. 97–102.]

The subject of detailed investigation here is the 3-line minors of the 2nd compound of $|a_1b_2c_3d_4|$, and the essence of the paper is to be found in the following classification and census of the said minors, 400 in number:—

Irresolvable like	$\Big\| \, \|a_1b_2\| \; \|a_1c_3\| \; \|b_2d_4\| \, \Big\|$	144
Resolvable like	$\Big\| \, \|a_1b_2\| \; \|a_1d_3\| \; \|b_2c_3\| \, \Big\| = \|a_1b_2c_3\| \cdot \|a_1b_2d_3\|$	96
,, ,,	$\Big\| \, \|a_1b_2\| \; \|a_1c_3\| \; \|b_1d_4\| \, \Big\| = \|a_1b_2\| \cdot \|a_1b_2c_3d_4\|$	96
,, ,,	$\Big\| \, \|a_1b_2\| \; \|a_1c_3\| \; \|b_2c_3\| \, \Big\| = \|a_1b_2c_3\|^2$	16
,, ,,	$\Big\| \, \|a_1b_2\| \; \|a_1c_3\| \; \|a_1d_4\| \, \Big\| = a_1^2 \, \|a_1b_2c_3d_4\|$	16
Vanishing like	$\Big\| \, \|a_1b_2\| \; \|a_1c_3\| \; \|b_1c_4\| \, \Big\|$	32.

USAI, G. ($1917/_4$)

[Una questione di analisi combinatoria. *Giornale di Mat.*, lv. pp. 127–138.]

This follows up the work of M. Pascal's paper of 1914, but the interest attaching to it is almost wholly non-determinantal.

MUIR, T. (1918$^{10}/_2$)

[Theorems connected with minors of an m-by-m^2 array. *Messenger of Math.*, xlvii. pp. 184–190.]

The fundamental theorem here is: *If the* m^2 *columns of an* m-*by*-m^2 *array be separated into* m + 1 *sets, namely,* m *sets of* m − 1 *columns each and a single set of* m *columns, then the determinant whose* (r, s)th *element is the determinant formed from the* rth *set and the* sth *column of the last set has the determinant of the last set for a factor, the cofactor being the compound determinant formed from the first* m *sets as the former compound determinant was formed from all the sets.* For example, when n is 4, and $|\, p, q, t, u\,|$ is used to stand for the determinant whose columns are the p^{th}, q^{th}, t^{th}, u^{th} columns of the array

$$\begin{array}{cccccccccccccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 & k_1 & l_1 & x_1 & y_1 & z_1 & w_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 & g_4 & h_4 & i_4 & j_4 & k_4 & l_4 & x_4 & y_4 & z_4 & w_4 \end{array}$$

we have

$$\begin{vmatrix} |1,\ 2,\ 3, 13| & |1,\ 2,\ 3, 14| & |1,\ 2,\ 3, 15| & |1,\ 2,\ 3, 16| \\ |4,\ 5,\ 6, 13| & |4,\ 5,\ 6, 14| & |4,\ 5,\ 6, 15| & |4,\ 5,\ 6, 16| \\ |7,\ 8,\ 9, 13| & |7,\ 8,\ 9, 14| & |7,\ 8,\ 9, 15| & |7,\ 8,\ 9, 16| \\ |10, 11, 12, 13| & |10, 11, 12, 14| & |10, 11, 12, 15| & |10, 11, 12, 16| \end{vmatrix}$$

$$= |13, 14, 15, 16| \begin{vmatrix} |1, 2, 3, 10| & |1, 2, 3, 11| & |1, 2, 3, 12| \\ |4, 5, 6, 10| & |4, 5, 6, 11| & |4, 5, 6, 12| \\ |7, 8, 9, 10| & |7, 8, 9, 11| & |7, 8, 9, 12| \end{vmatrix}.$$

The proof rests on the condensation-theorem of 1905, which in turn itself receives development: for example, its simplest case

$$\begin{vmatrix} |a_1b_2| & |c_1d_3| & |e_2f_3| \end{vmatrix} = \begin{vmatrix} |a_1b_2e_3| & |a_1b_2f_3| \\ |c_1d_2e_3| & |c_1d_2f_3| \end{vmatrix}$$

is shown to lead to

$$\begin{vmatrix} |a_1b_2| & |c_1d_3| & |e_2f_4| \end{vmatrix} + \begin{vmatrix} |a_1b_2| & |c_1d_4| & |e_2f_3| \end{vmatrix}$$

$$= \begin{vmatrix} |a_1b_2e_3| & |a_1b_2f_4| \\ |c_1d_2e_3| & |c_1d_2f_4| \end{vmatrix} + \begin{vmatrix} |a_1b_2e_4| & |a_1b_2f_3| \\ |c_1d_2e_4| & |c_1d_2f_3| \end{vmatrix},$$

and thence to another with *three* compound determinants on either side, and finally to another with *four*, the complete set forming *a series of relations connecting three-line minors of the second compound of* $|\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4\mathrm{e}_5\mathrm{f}_6|$ *with two-line minors of the third compound.* In the concluding paragraph the minors of another compound are also brought into the circle of relationships.

WHITTAKER, E. T. (1918$^{10}/_{9}$)

[On determinants whose elements are determinants. *Proceed. Edinburgh Math. Soc.*, xxxvi. pp. 107–115.]

The fresh idea here brought forward is interesting and suggestive, namely, the possibility of expressing certain compound determinants as determinants that are not compound, but have elements of the ordinary monomial kind; for example,

$$\begin{Vmatrix} |a_1b_2e_3| & |a_1b_2f_3| \\ |c_1d_2e_3| & |c_1d_2f_3| \end{Vmatrix} \quad \text{or} \quad \begin{Vmatrix} |a_1b_2| & |a_1b_3| & |a_2b_3| \\ |c_1d_2| & |c_1d_3| & |c_2d_3| \\ |e_1f_2| & |e_1f_3| & |e_2f_3| \end{Vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_1 & c_2 & c_3 \\ \cdot & \cdot & \cdot & d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 & e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 & f_1 & f_2 & f_3 \end{vmatrix}.$$

The types found to be so expressible are those of Muir's second paper of 1905, and the non-compound now shown to give rise to them is of the form

$$\begin{vmatrix} \{P_1\} & \{0\} & \{0\} & \cdots & \{0\} & \{0\} \\ \{0\} & \{P_2\} & \{0\} & \cdots & \{0\} & \{0\} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \{0\} & \{0\} & \{0\} & \cdots & \{0\} & \{P_m\} \\ \{P_{m+1}\} & \{P_{m+1}\} & \{P_{m+1}\} & \cdots & \{P_{m+1}\} & \{P_{m+1}\} \end{vmatrix},$$

where the $\{P\}$'s are m-by-$(m+1)$ arrays of any elements and the

$\{0\}$'s are m-by-$(m+1)$ arrays of zero elements, so that the order of the determinant is the $(m^2+m)^{\text{th}}$, the number of variables involved is $m^2(m+1)^2$, and the orders of the two compound determinants are the m^{th} and $(m+1)^{\text{th}}$ respectively. The transformation from the non-compound determinant to either of the others is effected by direct application of Laplace's expansion-theorem.

As a deduction there is given a theorem which may be looked on as a generalization (from 1 to p) of the "unextended" case of Bazin's of 1851, namely: *The determinant (hybrid adjugate, say) whose rows are the first* p *rows of the adjugate of* A *and the last* n — p *rows of the adjugate of* B *is equal to*

$$B^{n-p-1}\begin{vmatrix} (A_1, B_1) & (A_1, B_2) & \dots & (A_1, B_p) \\ (A_2, B_1) & (A_2, B_2) & \dots & (A_2, B_p) \\ \cdot & \cdot & \cdot & \cdot \\ (A_p, B_1) & (A_p, B_2) & \dots & (A_p, B_p) \end{vmatrix}$$

where n *is the order of* A *and* B, *and* $(\mathrm{A}_r, \mathrm{B}_s)$ *denotes the determinant got from* A *by replacing its* r^{th} *row by the* s^{th} *row of* B.

MUIR, T. (1918¹/₁₂)

[Note on the determinant of the primary minors of a special set of $(n-1)$-by-n arrays. *Proceed. R. Soc. Edinburgh*, xxxix. pp. 35–40.]

When n is 4 the arrays in question are

$$\begin{Vmatrix} 1 & a+b+c & ab+ac+bc & abc \\ 1 & e+f+g & ef+eg+fg & efg \\ 1 & i+j+k & ij+ik+jk & ijk \end{Vmatrix}, \quad \begin{Vmatrix} 1 & b+c+d & bc+bd+cd & bcd \\ 1 & f+g+h & fg+fh+gh & fgh \\ 1 & j+k+l & jk+jl+kl & jkl \end{Vmatrix}$$

$$\begin{Vmatrix} 1 & c+d+e & \dots \\ 1 & g+h+i & \dots \\ 1 & k+l+a & \dots \end{Vmatrix}, \quad \begin{Vmatrix} 1 & d+e+f & \dots \\ 1 & h+i+j & \dots \\ 1 & l+a+b & \dots \end{Vmatrix}$$

and the associated determinant referred to, Θ say, has for its $(r, s)^{\text{th}}$ element the minor got from the r^{th} array by deleting the s^{th} column. On using the condensation-theorem of 1905 (see

above, pp. 222–3) Θ is changed into a 3-line compound whose $(1, 1)^{\text{th}}$ element is

$$\begin{vmatrix} 1 & a+b+c & \dots \\ 1 & b+c+d & \dots \\ 1 & f+g+h & \dots \\ 1 & i+j+k & \dots \end{vmatrix}.$$

The fundamental property of determinants of this type is next given, so as to aid in the evaluation of Θ, and especially to ascertain under what circumstances Θ vanishes. Of the properties of Θ thus reached one of the most interesting is that when $g, l, i \equiv d, c, e$, its value is expressible as a product of eighteen differences, namely,

$$\begin{aligned} &(k-h)(d-j)(f-j)(a-e)(b-e)(c-e) \\ \cdot\, &(k-b)(d-a)(e-a)(d-c)(h-c)(e-c) \\ \cdot\, &(h-e)(c-f)(b-f)(j-d)(k-d)(c-d). \end{aligned}$$

MUIR, T. ($1919^{15}/_{1}$)

[Note on compound determinants expressible as simple determinants. *Quart. Journ. of . . . Math.*, xlviii. pp. 379–384.]

This note follows up Whittaker's of 1918. The two results arrived at in it may for comparison's sake be stated thus: *If* $\{m, m+1\}$ *denote an* m-*by*-(m + 1) *array of any elements, and* $\{0\}$ *an* m-*by*-(m + 1) *array of zeros, the non-compound determinant of the* $(pm + p)^{th}$ order

$$\begin{vmatrix} \{m, m+1\}_1 & \{0\} & \{0\} & \dots & \{0\} & \{0\} \\ \{0\} & \{m, m+1\}_2 & \{0\} & \dots & \{0\} & \{0\} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \{0\} & \{0\} & \{0\} & \dots & \{0\} & \{m, m+1\}_p \\ \{p, m+1\}_{p+1} & \{p, m+1\}_{p+1} & \{p, m+1\}_{p+1} & \dots & \{p, m+1\}_{p+1} & \{p, m+1\}_{p+1} \end{vmatrix}$$

is equal to a p-*line compound determinant with* (m + 1)-*line elements and the similar determinant of the* $(pm)^{th}$ *order*

$$\begin{vmatrix} \{m, m+1\}_1 & \{0\} & \{0\} & \dots & \{0\} & \{0\} \\ \{0\} & \{m, m+1\}_2 & \{0\} & \dots & \{0\} & \{0\} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \{0\} & \{0\} & \{0\} & \dots & \{m, m+1\}_{p-1} & \{0\} \\ \{m, m+1\}_p & \{m, m+1\}_p & \{m, m+1\}_p & \dots & \{m, m+1\}_p & \{m, -p+m+1\}_p \end{vmatrix}$$

is equal to a p-*line compound determinant with* m-*line elements.* In the paper, however, a slightly different notation for the arrays makes greater definiteness possible: for example, when p is 3 we have

$$\begin{vmatrix} A(m, m+1) & \{0\} & \{0\} \\ \{0\} & B(m, m+1) & \{0\} \\ \{0\} & \{0\} & C(m, m+1) \\ Z(3, m+1) & Z(3, m+1) & Z(3, m+1) \end{vmatrix} = (-1)^m \begin{vmatrix} A_{+1} & A_{+2} & A_{+3} \\ B_{+1} & B_{+2} & B_{+3} \\ C_{+1} & C_{+2} & C_{+3} \end{vmatrix}$$

where A_{+r} stands for the determinant got by affixing to the array A the r^{th} row of the last array Z; and

$$\begin{vmatrix} A(m, m+1) & \{0\} & \{0\} \\ \{0\} & B(m, m+1) & \{0\} \\ C(m, m+1) & C(m, m+1) & C(m, -3+m+1) \end{vmatrix} = (-1)^{m-1} \left| A_{-1} B_{-2} C_{-3} \right|,$$

where A_{-r} stands for the determinant got from the array A by deleting the r^{th} column, and $C(m, -r+m+1)$ stands for the array got from C by deleting the first r columns.

The original suggesting theorem is the case of the first above when p is taken equal to m, and the case of the second when p is taken equal to $m+1$, the compound determinant got from the first being of the m^{th} order with $(m+1)$-line elements, and that got from the second being of the $(m+1)^{\text{th}}$ order with m-line elements.

MUIR, T. (1919 15/7)

[The eliminant of two binary quantics with determinantal coefficients. *Messenger of Math.*, xlix. pp. 37–41.]

The curious result here lit upon is that the eliminant of

$$\left.\begin{aligned} |a_1b_2c_3|\,x^3 - |a_1b_2c_4|\,x^2 + |a_1b_3c_4|\,x - |a_2b_3c_4| &= 0 \\ |a_2b_3c_4|\,x^3 - |a_2b_3c_5|\,x^2 + |a_2b_4c_5|\,x - |a_3b_4c_5| &= 0 \end{aligned}\right\}$$

mimics Bezout's eliminant of

$$\left.\begin{aligned} a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 &= 0 \\ b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5 &= 0 \end{aligned}\right\},$$

being

$$\begin{vmatrix} |123| & |124| & |134| & |234| \\ |124| & |134|+|125| & |234|+|135| & |235| \\ |134| & |234|+|135| & |235|+|145| & |245| \\ |234| & |235| & |245| & |345| \end{vmatrix}.$$

In seeking for an explanation of this, the additional result is obtained that the eliminant of

$$\left\|\begin{matrix} a_1y-l_1x & a_2y-l_2x & a_3y-l_3x & a_4y-l_4x \\ b_1y-m_1x & b_2y-m_2x & b_3y-m_3x & b_4y-m_4x \\ c_1y-n_1x & c_2y-n_2x & c_3y-n_3x & c_4y-n_4x \end{matrix}\right\| = 0$$

is

$$\begin{vmatrix} |a_1b_2c_3l_4| & |a_1b_2n_3l_4|+|a_1m_2c_3l_4| & |a_1m_2n_3l_4| \\ |a_1b_2c_3m_4| & |a_1b_2n_3m_4|+|l_1b_2c_3m_4| & |l_1b_2n_3m_4| \\ |a_1b_2c_3n_4| & |a_1m_2c_3n_4|+|l_1b_2c_3n_4| & |l_1m_2c_3n_4| \end{vmatrix}.$$

MUIR, T. (1919[24]/[9])

[Additional note on the resolvability of the minors of a compound determinant. *Transac. R. Soc. S. Africa*, viii. 229–233.]

Elimination by the "method of instalments" here furnishes a suggestion for factorization. If, for example, the set of six homogeneous linear equations

$$\left.\begin{matrix} a_1u+a_2v+a_3w+a_4x+a_5y+a_6z=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_1u+f_2v+f_3w+f_4x+f_5y+f_6z=0 \end{matrix}\right\}$$

be taken, and u, v, w be eliminated from every subset of four, there are obtained 15 equations in x, y, z, and then from the $C_{15,3}$ triads of these are got 455 eliminants like

$$\begin{vmatrix} |a_1b_2c_3d_4| & |a_1b_2c_3d_5| & |a_1b_2c_3d_6| \\ |a_1b_2c_3e_4| & |a_1b_2c_3e_5| & |a_1b_2c_3e_6| \\ |a_1b_2c_3f_4| & |a_1b_2c_3f_5| & |a_1b_2c_3f_6| \end{vmatrix},$$

of which the true eliminant $|a_1b_2c_3d_4e_5f_6|$, or Δ_6 say, must be a factor. Full factorization of a large number of 3-line minors of the 4th compound of Δ_6 is thus seen to be rendered easy; and, the requisite work having been performed, the following census of results is given: *Of* 455 *three-line minors of the fourth compound of* Δ_6

15 *minors are of the type* $\begin{vmatrix} |a_1b_2d_3| & |a_1b_2c_3| \\ |d_1e_2f_3| & |c_1e_2f_3| \end{vmatrix} \cdot \Delta_6,$

180 *minors are of the type* $|a_1b_2c_3| \cdot |a_1d_2e_3| \cdot \Delta_6,$

180 *minors are of the type* $|a_1b_2c_3| \cdot |a_1b_2d_3| \cdot \Delta_6,$

20 *minors are of the type* $|a_1b_2c_3|^2 \cdot \Delta_6,$

and 60 *minors are equal to* 0.

MUIR, T. ($1919^5/_{12}$)

[Note on the m^{th} compound of a determinant of the $(2m)^{\text{th}}$ order. *Messenger of Math.*, xlix. pp. 62–64.]

The result here reached is really a generalization connected with the compound just mentioned. When m is 2 it concerns a 4-by-9 array of elements, and is

$$\begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| & |a_2b_3| & |a_2b_4| & |a_3b_4| \\ |a_1c_2| & |a_1c_3| & |a_1c_4| & |a_2c_3| & |a_2c_4| & |a_3c_4| \\ |b_1c_2| & |b_1c_3| & |b_1c_4| & |b_2c_3| & |b_2c_4| & |b_3c_4| \\ |d_1e_2| & |d_1e_3| & |d_1e_4| & |d_2e_3| & |d_2e_4| & |d_3e_4| \\ |f_1g_2| & |f_1g_3| & |f_1g_4| & |f_2g_3| & |f_2g_4| & |f_3g_4| \\ |h_1k_2| & |h_1k_3| & |h_1k_4| & |h_2k_3| & |h_2k_4| & |h_3k_4| \end{vmatrix}$$

$$= \begin{vmatrix} |a_1b_2d_3e_4| & |a_1b_2f_3g_4| & |a_1b_2h_3k_4| \\ |a_1c_2d_3e_4| & |a_1c_2f_3g_4| & |a_1c_2h_3k_4| \\ |b_1c_2d_3e_4| & |b_1c_2f_3g_4| & |b_1c_2h_3k_4| \end{vmatrix}.$$

An interesting stage in the course of degeneration to the compound is arrived at by putting the e's equal to the c's, and the g's equal to the b's.

CHAPTER VIII

Recurrents, from 1900 to 1919

In the period now to be considered the interest taken in Recurrents did not slacken among workers: on the contrary, the number of contributions is found to have been about a fourth more than in the period 1880–1898. At the same time the nature and quality of them are very little altered: for example, almost a half of them belong to the 'solved-problem' type. And many of those that are not of this class show equally little fitness for uniting with one another towards the formation of bigger areas or longer chains of algebraical truth.

For the sake of readier reference it may be well to note that a small sub-class, persymmetric recurrents, are not to be sought for under persymmetrics.

'Anon.' (1896)

[Opgave 72. *Nyt Tidsskrift f. Mat.*, B, vii. p. 79.]

The determinant evaluated here is, like d'Ovidio's of 1863 (*Hist.*, iii. p. 217), a bordered persymmetric recurrent, the result for the 4th order being

$$\begin{vmatrix} \frac{a}{1!} & 1 & \cdot & \cdot \\ \frac{b}{2!} & \frac{1}{1!} & 1 & \cdot \\ \frac{c}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 \\ \frac{d}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} = \frac{1}{4!}\left\{4a - 6b + 4c - d\right\}.$$

GAVRILOVITCH, B. (1900)

[On the analytic expression of certain functions (In Serbian). *Glasa Srpske Kralj. Akad.* (Belgrade), lxi. pp. 55–68.]

The main contribution here is essentially the rediscovery of Faure's theorem of 1855 regarding the quotient of two power-series (*Hist.*, ii. pp. 212–214): the fresh exposition with its illustrations is nevertheless of value.

STUDNIČKA, F. J. (1900⁹/₃)

[Ueber ein Analogon der Euler'schen Zahlen. *Sitzungsb. . . . Akad. d. Wiss.* (Prag), No. 9, 8 pp.]

The new numbers referred to, named A's as companions of the E's, are defined by the equality

$$\tan x = \sum_{n=0}^{n=\infty} A_{2n+1} \frac{x^{2n+1}}{(2n+1)!}.$$

Since, however, it is known that $A_{2n-1} = \dfrac{2^{2n}(2^{2n}-1)}{2n} B_n$, a separate designation seems hardly necessary. This relation probably also accounts for the fact that the recurrents obtained for E_n and A_{2n-1} are Hammond's of 1875 (*Hist.*, iii. p. 233). A second recurrent obtained for E_n involves the A's, and a companion for A_{2n-1} involves the E's.

GAVRILOVITCH, B. (1900²⁰/₄)

[On Bernoulli's and Euler's numbers (In Serbian). *Glasa Srpske Kralj. Akad.*, lxiii. pp. 131–142.]

The contents of this are in considerable part the results of Studnička's paper just reported on; but the methods are more interesting and the exposition is improved.*

* On the authority of Pascal he attributes the recurrent obtained for B_n by Siacci in 1865 (*Hist.*, iii. p. 222) to a later author.

ANDEREGG, F. ($1901^1/_2$)

[Question 129. *American Math. Monthly*, viii. p. 54: ix. pp. 11–13.]

The determinant appearing here, in connection with the summation of the series

$$1+\frac{2^m}{2!}+\frac{3^m}{3!}+\ldots$$

is

$$\begin{vmatrix} 1 & -1 & . & . & . & . & . \\ 1 & 1 & -1 & . & . & . & . \\ 1 & 2 & 1 & -1 & . & . & . \\ 1 & 3 & 3 & 1 & -1 & . & . \\ . & . & . & . & . & . & . \end{vmatrix}_m \quad \text{or } D_m \text{ say,}$$

whose values for the first six orders are

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ \ldots$$

and whose recurrent law of formation is

$$D_{m+1}=D_m+(m)_1D_{m-1}+(m)_2D_{m-2}+\ \ldots+(m)_m.$$

STUDNIČKA, F. J. ($1901^{26}/_4$)

[Ueber die independente Zerlegung von gebrochenen algebraischen Funktionen in Partialbrüche durch sphenoidale Derivationsdeterminante. *Sitzungsb. . . . Ges. d. Wiss.* (Prag) No. 18, 5 pp.; also, more fully in *Časopis pro pěstování math. a. fys.*, xxxi. pp. 1–10.]

The determinant utilized for the purpose indicated in the title is Spottiswoode's representative of the n^{th} differential-coefficient of a quotient (*Hist.*, ii. pp. 210–211).

Note should be taken in passing of the use which the writer makes of the terms ‘recurrent’ and ‘sphenoidal’.

ESTANAVE, E. ($1901^1/_5$)

[Question 2078. *L'Intermédiaire des Math.*, viii. p. 109: xxiii. p. 29.]

The interesting equality given here remains still unproved, although proof has more than once been called for. Doubtless the problem has been made unnecessarily forbidding by the presence of an irrelevant factor common to both members. Removing this and modifying the disposition of signs we have for the essential equality

$$(-1)^m 2(1-2^{2m-1}) \times \begin{vmatrix} \frac{1}{2!} & 1 & . & \dots & . \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & . \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \dots & . \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(2m+1)!} & \frac{1}{(2m)!} & \frac{1}{(2m-1)!} & \dots & \frac{1}{2!} \end{vmatrix} = \begin{vmatrix} \frac{1}{3!} & 1 & . & \dots \\ \frac{1}{5!} & \frac{1}{3!} & 1 & \dots \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_m$$

both determinants being persymmetric recurrents. Further, as the recurrent on the left is known (*Hist.*, iii. p. 222) to be equal to the m^{th} Bernoulli number multiplied by $(-1)^{m+1}/(2m)!$ we may view ourselves as being required to show that the recurrent on the right is equal to

$$\frac{2m-2}{(2m)!}B_m,$$

a task of no great difficulty. If, however, the problem be held to be the evolving of the one recurrent out of the other, the task is harder. Interesting in connection with this latter view is the fact that the right-hand recurrent is the minor got from the left-hand one by deleting the odd-numbered rows and the even-numbered columns.

MIGNOSI, G. (1902)

[Un problema sulla partizione dei numeri. *Periodico di Mat.*, xviii. pp. 117–123.]

The k-line recurrent which makes its appearance here is persymmetric save for the commencing diagonal which is $-1, -2, -3, \ldots,$ the other elements being each connected with the sum of all the divisors of an integer that are not above a given limit. The evaluation of a recurrent of the same form but having the elements in its first column unspecialized has already been given (*Hist.*, iii. p. 228).

ESTANAVE, E. (1902³/₁₂), (1903/₂)

[Sur les coefficients des développements en séries de tang x, séc x et d'autres fonctions. Leur expression à l'aide d'un déterminant unique. *Bull. . . . Soc. Math. de France*, xxxi. pp. 203–208.]

[Question 2515. *L'Intermédiaire des Math.*, x. p. 35: xxi. p. 108.]

The determinant referred to in the title is the equivalent of A_n in the expansion

$$\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = A_0 - A_1\frac{x}{1} + A_2\frac{x^2}{2!} - A_3\frac{x^3}{3!} + \ldots$$

Since, as is pointed out,

$$A_{2p-1} = \frac{2^{2p}(2^{2p}-1)}{2p}B_p \quad \text{and} \quad A_{2p} = E_p,$$

its law of formation closely resembles Hammond's of 1875 for B_p and E_p (*Hist.*, iii. pp. 233–234). It is, however, more complicated than either, because A_n has of course to include both A_{2p-1} and A_p.

The matter set for proof by the same writer is the equality of a p-line recurrent to a multiple of a $(p+1)$-line recurrent. It is, however, too defectively stated to be of use.

ROE, E. D. ($1904/_2$)

[On complete symmetric functions. *American Math. Monthly*, xi. pp. 156–163, 179–184.]

Part of this (pp. 156–163) is a valuable compendium of over thirty relations concerned with the simpler types of symmetric functions, and among them are found the determinantal relations, connecting the so-called c's, s's and alephs, known since Brioschi's time.

BOURLET, C. ($1904^1/_5$)

[Question 1998. *Nouv. Annales de Math.*, (4) iv. p. 240: v. pp. 91–93.]

The result here established,

$$\begin{vmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdots \\ 1 & (3)_2 & 1 & \cdot & \cdot & \cdots \\ 1 & (4)_2 & (4)_3 & 1 & \cdot & \cdots \\ 1 & (5)_2 & (5)_3 & (5)_4 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}_m = m,$$

is closely connected with a determinant of Tirelli's which we showed to be transformable into a persymmetric continuant (*Hist.*, iii. p. 459).

EPSTEIN, P. ($1904/_{12}$)

[Aufgaben und Lehrsätze, Nr. 117. *Archiv d. Math. u. Phys.*, (3) viii. pp. 329–330: ix. pp. 189–191.]

The question here is: In how many ways can a number which is the product of m prime numbers be expressed as a product of mutually prime numbers? And the answer is

$$\begin{vmatrix} 2 & -1 & \cdot & \cdot & \cdots \\ \frac{1}{1!} & 2 & -2 & \cdot & \cdots \\ \frac{1}{2!} & \frac{1}{1!} & 2 & -3 & \cdots \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}_{m-1}$$

the values of the first six orders being

$$2,\ 5,\ 15,\ 52,\ 203,\ 877,\ \ldots$$

A very interesting question is raised in view of the identity of the values of this recurrent with those of Anderegg's of 1901 (see above).

PASCAL, E. ($1907/_2$)

[I determinanti ricorrenti e le loro proprietà. *Rendic. . . . Istituto Lombardo* . . ., (2) xl. pp. 293–305.]

This paper owes its origin to the discovery by P. Burgatti of the series of vanishing recurrents

$$\begin{vmatrix} (m)_1 & 2 \\ (m)_2 & (m-1)_1 \end{vmatrix}, \quad \begin{vmatrix} (m)_1 & 2 & \cdot \\ (m)_3 & (m-1)_2 & 2 \\ (m)_4 & (m-1)_3 & (m-3)_1 \end{vmatrix},$$

$$\begin{vmatrix} (m)_1 & 2 & \cdot & \cdot \\ (m)_3 & (m-1)_2 & 2 & \cdot \\ (m)_5 & (m-1)_4 & (m-3)_2 & 2 \\ (m)_6 & (m-1)_5 & (m-3)_3 & (m-5)_1 \end{vmatrix}, \ldots$$

whose law of formation is most readily seen by viewing them as got by bordering another of simpler form: for example, the next unbordered is

$$\begin{matrix} 2 & \cdot & \cdot & \cdot \\ (m-1)_2 & 2 & \cdot & \cdot \\ (m-1)_4 & (m-3)_2 & 2 & \cdot \\ (m-1)_6 & (m-3)_4 & (m-5)_2 & 2 \end{matrix}$$

and the bordering column and row are

$$m_1\ m_3\ m_5\ m_7\ m_8 \quad \text{and} \quad m_8\ (m-1)_7\ (m-3)_5\ (m-5)_3\ (m-7)_1.$$

Pascal renders a double service by first establishing the additional equality

$$\begin{vmatrix} \frac{1}{1!} & 2 & . & \dots & . \\ \frac{1}{3!} & \frac{1}{2!} & 2 & \dots & . \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \dots & . \\ . & . & . & . & . \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \dots & 2 \\ \frac{1}{(2n)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \dots & \frac{1}{1!} \end{vmatrix} = 0$$

and then deriving Burgatti's from it. The mode of derivation is based on showing that an n-line recurrent is multiplied by $x_1x_2 \dots x_n$ if the individual elements in it be each multiplied by an appropriate product of x's. To bring this lemma into touch with allied work of earlier date it is best for us to formulate the following generalized double theorem: *If the elements of the last row of an* n-*line determinant be multiplied by*

$$x_1x_2 \dots x_n, \quad x_2x_3 \dots x_n, \quad x_3x_4 \dots x_n, \quad \dots, \quad x_{n-1}x_n, \quad x_n$$

respectively, the elements of the row before it by

$$x_1x_2 \dots x_{n-1}, \; x_2x_3 \dots x_{n-1}, \; x_3x_4 \dots x_{n-1}, \quad \dots, \; x_{n-1}, \; 1$$

respectively, and so on with the other rows, the multipliers for the r^{th} *row being* x_r *times those of the* $(r-1)^{th}$, *the determinant is thereby multiplied by* $x_1x_2 \dots x_n$: *if, on the other hand, we begin with the above second set of multipliers and proceed upwards as before, the determinant is in substance unaltered.* Now when in the former half of this the determinant is made a recurrent we obtain Pascal's lemma; and when the x's in the latter half, which is Laisant's theorem of 1901, are all made equal we obtain Fürstenau's result of 1879 (*Hist.*, iii. p. 82). The value of the mode of procedure is evidenced by another curious and interesting equality, namely,

$$\begin{vmatrix} \frac{1}{2!} & 2 & \cdot & \cdots & \cdot \\ \frac{1}{4!} & \frac{1}{2!} & 2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \frac{1}{(2n-6)!} & \cdots & 2 \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \frac{1}{(2n-5)!} & \cdots & \frac{1}{1!} \end{vmatrix}$$

$$= 2^n \begin{vmatrix} \frac{1}{3!} & 1 & \cdot & \cdots & \cdot \\ \frac{1}{5!} & \frac{1}{3!} & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \frac{1}{(2n-5)!} & \cdots & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2!} \end{vmatrix}$$

where it almost looks as if the last rows ought to be interchanged.

Prefaced to these results is a short introduction on the more general determinant

$$\begin{vmatrix} a_{10} & -t & \cdot & \cdot & \cdots \\ a_{20} & a_{21} & -t & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n0} & a_{n1} & a_{n2} & a_{n3} & \cdots \end{vmatrix}.$$

This he defines as a 'recurrent', wisely avoiding Studnička's proposed use of both this term and the term 'sphenoidal'. It may not be out of place to add that in 1910 we followed in Pascal's footsteps, merely giving the word a slightly extended meaning.

PASCAL, E. (1907/3)

[I determinanti ricorrenti e i nuovi numeri pseudoeuleriani *Rendic. . . . Ist. Lombardo . . .*, (2) xl. pp. 461–475.]

[Una formola sui coefficienti polynomiali e su di un determinante ricorrente. *Periodico di Mat.*, xxii. pp. 241–245.]

So far as determinants are concerned there is little to note in the first of these papers. The new numbers occupy the same places in the expansion of $1/(2 - \cosh x)$ as the Eulerian numbers occupy in the expansion of $1/\cos x$: and the recurrents corresponding to the latter numbers (*Hist.*, iii. pp. 233, 236) give those corresponding to the former on their diagonals of units being changed in sign.

The second note contains an alternative proof of the author's companion to Burgatti's vanishing recurrent.

PASCAL, E. (1907 28/4)

[I nuovi numeri pseudo-tangenziali. *Rendic. del Circolo Mat.* (Palermo), xxiii. pp. 358–366.]

The so-called 'tangential' numbers implicitly referred to in the title are those connected with the Bernoullian B_{2n} by the multiplier $2^{2n}(2^{2n} - 1)/2n$, and first expressed in determinant form by Hammond in 1875 (*Hist.*, iii. p. 233). From this form the writer derives the equivalent

$$(2n-1)!\begin{vmatrix} \frac{1}{1!} & 1 & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \cdot & \cdots & \cdot \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \frac{1}{(2n-5)!} & \cdots & \frac{1}{2!} \end{vmatrix}$$

and then proceeds to show that, if instead of $\tan x$ we take $\sinh x/(2 - \cosh x)$ as the generating function, the corresponding

change in his determinant is that the diagonal of units is made negative.

MIGNOSI, G. ($1907/_5$)

[Sulla equazione lineare indeterminata. *Periodico di Mat.*, xxiii. pp. 173–176.]

The recurrent which appears here is of the same type as that already mentioned in the same author's paper of 1902, and like it is connected with the Theory of Integers. A curious special case of it is

$$\begin{vmatrix} m & -1 & \cdot & \cdot & \cdots \\ m & m & -2 & \cdot & \cdots \\ m & m & m & -3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}_n = m(m+1)(m+2)\ldots(m+n-1).$$

MUIR, T. ($1907/_{12}$)

[A special determinant having the $(r, s)^{\text{th}}$ element equal to zero when $s > r + 1$. *Transac. S. African Philos. Soc.*, xviii. pp. 369–377.]

The special recurrent referred to is for the 2^{nd}, 3^{rd}, 4^{th}, ... orders

$$\begin{vmatrix} b_1 & a_0 \\ a_1 & b_1 \end{vmatrix}, \quad \begin{vmatrix} b_1 & a_0 & \cdot \\ b_2 & a_1 & a_0 \\ a_2 & b_2 & b_1 \end{vmatrix}, \quad \begin{vmatrix} b_1 & a_0 & \cdot & \cdot \\ b_2 & a_1 & a_0 & \cdot \\ b_3 & a_2 & a_1 & a_0 \\ a_3 & b_3 & b_2 & b_1 \end{vmatrix}, \quad \ldots$$

and the writer, following Pascal, seeks to ascertain the conditions which will entail evanescence. These for the n-line recurrent he finds to be the n equations

$$(a_0 \between a_1) - (b_1 \between b_1) = 0,$$
$$(a_0, a_1 \between a_2, a_1) - (b_1, b_2 \between b_2, b_1) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;$$

and as they are satisfied if the a's be made the even-numbered members and the b's the odd-numbered members of the series

$$\frac{1}{1!},\ \frac{1}{2!},\ \frac{1}{3!},\ \ldots,\ \frac{1}{(2n)!},$$

Pascal's vanishing recurrent is at once obtained. Further—and this more noteworthy—a proof is given that if one such recurrent be found, a companion to it is always ready to hand. Here, for example, the companion is

$$\begin{vmatrix} 1 & \frac{1}{2!} & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{3!} & \frac{1}{4!} & \frac{1}{2!} & \cdots & \cdot & \cdot \\ \frac{1}{5!} & \frac{1}{6!} & \frac{1}{4!} & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{(2n-3)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2!} & \cdot \\ \frac{1}{(2n-1)!} & \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \cdots & \frac{1}{3!} & 1 \end{vmatrix} = 0$$

TANTURRI, A. (1907)

[Dalla formola di Pascal a quella di Bernoulli sulle somme delle potenze simili dei primi n numeri. *Periodico di Mat.*, xxiii. pp. 80–83.]

The point of interest for us in this is that the deduction mentioned is made from the recurrent, and that this is obtained at the outset exactly as by Siacci in 1865 (*Hist.*, iii. pp. 221–222).

SINIGALLIA, L. (1907/$_{6,\,7}$)

[Sui nuovi numeri pseudo-euleriani del Prof. Pascal. *Rendic. del Circolo Mat.* (Palermo), xxiv. pp. 223–228.]

[Una estensione dei numeri bernoulliani. *Rendic. del Circolo Mat.* (Palermo), xxv. pp. 20–35.]

A glance at the results of Pascal's paper referred to (see above) prepares one to expect that for every known theorem involving Eulerian numbers (E) there must almost of necessity be a corresponding theorem in which these are replaced by the analogous new numbers (E'). Such an expectation the first of Sinigallia's papers amply justifies. Perhaps the most interesting conclusion reached in it is that

$$\text{since } E_{2n} = \sum_{\nu=1}^{\nu=n} (-1)^{\nu+1}(2n)_{2\nu}E_{2(n-\nu)}.$$

$$\text{and } E'_{2n} = \sum_{\nu=1}^{\nu=n} (2n)_{2\nu}E'_{2(n-\nu)},$$

it can readily be shown that

$$E_{2n} + (-1)^n E'_{2n} = 2\sum_{i=1}^{i=n-1} (-1)^{i+1}(2n)_{2i}E'_{2i}E_{2(n-i)},$$

a conjoint recurrence-formula of a type whose importance we have already drawn attention to (*Hist.*, iii. p. 247).

The interest of the second paper for us lies in the fact that the extension arrived at is suggested by extending the determinant form of the numbers in question.

DICKSON, L. E. (1907/$_8$): 'ANON.' (1909): FONTENÉ, G. (1910^1/$_5$)

[Question 288. *American Math. Monthly*, xiv. p. 160: xv. p. 35.]

[Question 16771. *Educ. Times*, lxii. p. 513: lxiii. p. 133: or *Math. from Educ. Times*, (2) xviii. pp. 51–53.]

[Question 2151. *Nouv. Annales de Math.*, (4) x. p. 239.]

The recurrents brought forward here are Weihrauch's of 1874, D'Ovidio's of 1863, and Glaisher's of 1876 (*Hist.*, iii. pp. 231–232, 217–219, 234).

HERNÁNDEZ, E. ($1908^1/_7$, $1908^1/_8$): AURIC, A. (1908)

[Question 3363. *L'Intermédiaire des Math.*, xv. pp. 75, 166–168, 283.]

[Question 16475. *Educ. Times*, lxi. p. 351: *Math. from Educ. Times*, (2) xv. pp. 90–92.]

The net outcome of the intercommunications in regard to the first of these questions is an alternative proof by Auric that the odd-ordered persymmetric recurrent with elements 1, 1/2!, 1/3!, . . . vanishes (*Hist.*, iii. p. 234), and a similar proof that the like is true of a new odd-ordered recurrent. In the latter case, however, there is established the interesting general result:

$$\begin{vmatrix} 1 & 1 & \cdot & \cdot & \cdots \\ \frac{1}{2!} & \frac{2}{2!} & 1 & \cdot & \cdots \\ \frac{1}{3!} & \frac{2^2}{3!} & \frac{2}{2!} & 1 & \cdots \\ \frac{1}{4!} & \frac{2^3}{4!} & \frac{2^2}{3!} & \frac{2}{2!} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n!} & \frac{2^{n-1}}{(n-1)!} & \frac{2^{n-2}}{(n-2)!} & \frac{2^{n-3}}{(n-3)!} & \cdots \frac{2}{2!} \end{vmatrix} = (-1)^{n-1} \cdot \frac{2\chi_n(0)}{n!},$$

where, but for the first column, the determinant is persymmetric, and where $\chi_n(z)$, defined by the equality

$$\frac{e^{xz}}{e^x + 1} = \Sigma \frac{\chi_n(z)}{n!} x^n,$$

is such that

$$\chi_{2n}(0) = 0.$$

The other question merely concerns two simple cases of Brioschi's recurrent of 1854 for the sum of the m^{th} powers of the roots of an equation (*Hist.*, ii. p. 211).

SALKOWSKI, E. (1908 24/9)

[Die n-te Ableitung eines Quotienten. *Archiv d. Math. u. Phys.*, (3) xiii. pp. 371–373.]

This is not at all a casual republication of Spottiswoode's result of 1853 (*Hist.*, ii. p. 210): the said result is capably discussed and established anew. The author also draws attention to two relevant passages unnoted by us, namely, one in Hoüel's *Cours de Calcul Inf.* i., and the other in *Crelle's Journ.*, civ. (1888), pp. 102–115. To these we may now add *Monatshefte f. Math. u. Phys.*, i. (1890), pp. 33–38.*

GODEAUX, L. (1909 2/9)

[Sur les déterminants récurrents du Prof. E. Pascal. *Rendic. del Circolo Mat.* (Palermo), xxix. pp. 261–264.]

Here the determinant

$$\begin{vmatrix} a_{11} & x+y & \cdot & \cdot & \cdots \\ a_{12} & a_{22} & x+y & \cdot & \cdots \\ a_{13} & a_{23} & a_{33} & x+y & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}_n, \text{ or say } \mu_n(x+y),$$

is expressed in two ways as a series arranged according to ascending powers of y. Unfortunately the coefficients in neither case are readily calculable, the one series being simply Taylor's

$$\mu_n(x+y) = \mu_n(x) + y\frac{d\mu_n(x)}{dx} + \tfrac{1}{2}y^2\frac{d^2\mu_n(x)}{d^2x} + \cdots$$

and the other being got by viewing all the elements as binomials $a_{11}+0$, $a_{12}+0$, . . . and expanding in terms of determinants with monomial elements. The deductions thence arrived at are thus not particularly noteworthy.

* In the *American Math. Monthly*, xvi. pp. 15–16, the determinant is retained in the second order.

MUIR, T. ($1910^{13}/_6$)

[The theory of recurrent determinants . . . up to 1860. *Proceed. R. Soc. of Edinburgh*, xxxi. pp. 304–310.]

The title of this, the first of our historical papers on recurrents, is not quite accurate, the earliest of the eight writings considered in it being Spottiswoode's of 1853, whereas, as we have pointed out elsewhere (*Hist.*, ii. p. 210), the form had made its appearance forty years earlier.

FERRARI, F. (1910): BOTTARI, A. (1912)

[Sulla somma delle potenze simili dei primi n numeri naturali. *Periodico di Mat.*, xxv. pp. 258–261.]

[Somma delle potenze simili di due quantità in funzione della loro somma e del loro prodotto. *Periodico di Mat.*, xxvii. pp. 104–120.]

Putting

$$1^m + 2^m + \ldots + n^m \equiv A_1 n^{m+1} + A_2 n^m + \ldots + A_{m+1} n$$

the first writer here finds that $A_1 = \dfrac{1}{m+1}$, $A_2 = -\dfrac{1}{2}$ and generally that

$$A_k = \frac{(-1)^{k+1}}{(m+1)m\ldots(m-k+2)} \cdot \begin{vmatrix} (m+1)_2 & (m)_1 & . & \ldots & . \\ (m+1)_3 & (m)_2 & (m-1)_1 & \ldots & . \\ . & . & . & . & . \\ (m+1)_k & (m)_{k-1} & (m-1)_{k-2} & \ldots & (m-k+3)_2 \end{vmatrix}.$$

The second writer correctly evaluates several recurrents connected with his main subject, not observing, however, that they are special cases of Gubler's of 1890 (*Hist.*, iv. p. 234).

KAPTEYN, W. (1912)

[Vraagstukken 46, 49. *Wiskundige Opgaven*, xi. pp. 119–122, 175–178.]

The recurrent here dealt with is the familiar

$$\begin{vmatrix} a_1 & 1 & \cdot & \cdots & \cdot \\ a_2 & a_1 & 2 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix} \quad \text{or } \Delta_n \text{ say,}$$

where the diagonal bordering on the zeros is 1, 2, . . ., $n-1$, and the other diagonals are persymmetric. The interesting property established is that

$$\frac{\partial \Delta_n}{\partial a_k} = (-1)^{k-1} \frac{n!}{(n-k)!k} \Delta_{n-k};$$

and following on this comes a like property for the determinant got by bordering Δ_n by the row b_1, 1, 0, 0, . . . and the column $b_1, b_2, \ldots, b_{n+1}$. Among the proofs given there is no purely determinantal one that is convincing and at the same time concise.

ROSS, C. M. (1912, 1914): ONO, T. (1913¹/₈)

[Question 17311. *Educ. Times*, lxv. p. 262.]

[Question 17770. *Educ. Times*, lxvii. p. 354: *Math. from Educ. Times*, (2) xxvii. p. 44, p. 63.]

[Question 4253. *L'Intermédiaire des Math.*, xxiv. pp. 7–9, 106–110.]

The first of these gives the generating function of the recurrent mentioned above under Anderegg (1901). The second is already known (*Hist.*, iii. p. 237, p. 460). The third, which is much the most interesting of the three, in that it requires a proof of the equality of two recurrents of different types, did not receive effectual attention until 1917. And even then recourse had to be had to the usually inattractive method of evaluating both members of the asserted equality,—a method, too, which had been open from the first, as one of the two determinants was

recognizable as Brioschi's evaluated persymmetric recurrent of 1858 (*Hist.*, iii. pp. 208–209).

POLVANI, G. (1913)

[Sopra le frazioni di Lambert. *Periodico di Mat.*, xxvii. pp. 241–266.]

The fractions referred to are those spoken of above as 'ascending continued fractions' (*Hist.*, iii. pp. 230–231). In studying their properties one of the instruments used is the appropriate recurrent, but the writer is only concerned with it in so far as it furthers his purposes.

HASSELT, G. v. (1914, 1914): WIJTHOFF, W. A. (1914)

[Vraagstukken 158, 180. *Wiskundige Opgaven*, xi. pp. 402–407, pp. 454–455.]

[Vraagstuk 159. *Wiskundige Opgaven*, xi. pp. 410–411.]

The first result established here is a relation between two kinds of simple symmetric functions, namely, the relation

$$\begin{vmatrix} -k & 1 & \cdot & \cdot & \cdots & \cdot \\ s_1 & a_1 & 1 & \cdot & \cdots & \cdot \\ s_2 & a_2 & a_1 & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ s_k & a_k & a_{k-1} & a_{k-2} & \cdots & a_1 \end{vmatrix} = 0,$$

where s_k is the sum of the k^{th} powers of the roots of the equation $x^n + a_1x^{n-1} + \ldots + a_n = 0$. None of the proofs given make use of the operation

$$\text{col}_1 + k\,\text{col}_2 + (k-1)\aleph_1\,\text{col}_3 + (k-2)\aleph_2\,\text{col}_4 + \ldots$$

suggested by Cherriman's relation of 1882 (*Hist.*, iv. p. 227).

The third question concerns Anderegg's problem of 1901, and the same determinant again makes its appearance.

The recurrent dealt with in the second is got by bordering the well-known p-line persymmetric recurrent equal to $1/p!$ (*Hist.*, iii. p. 217), the bordering column being $1, a_1, a_2, \ldots$

where $a_p = (n + p)_p/p!$ and the bordering row is 1, 1, 0, 0, . . . Its value is found to be $(-1)^p(n)_p/p!$

AIYAR, S. N. ($1914/_{12}$)

[Question 592. *Journ. Indian Math. Soc.*, vii. p. 148.]

The $(n + 1)$-line recurrent here is that whose r^{th} row is

$$(1 + a)^{r-1},\ 1,\ (r - 1)_1,\ (r - 1)_2,\ \ldots$$

It is such that $D_{n+1} = - aD_n$, and consequently equals $(-a)^n$.

CIPOLLA, M. ($1915^5/_2$)

[Determinanti della teoria dei numeri. *Atti dell' Accad. Gioenia* . . . (Catania), (5) viii. No. 12, 10 pp.]

We have already had, in H. J. S. Smith's axisymmetric determinant of 1876 (*Hist.*, iii. pp. 116–121), a conspicuous instance of an isolated determinantal result in the Theory of Integers being followed up by interesting generalizations (*Hist.*, iv. p. 118). Smith's determinant, it will be remembered, was an expression for $\psi(1) \cdot \psi(2) \ldots \psi(n)$ where $\psi(n)$ stands for the number of integers that are less than n and prime to n. The similarly originating result now is a determinant expression for $\psi(n)$ itself, namely, the recurrent

$$\begin{vmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & 3 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 10 & 4 & 2 & 1 & 1 & \cdot & \cdot & \cdot \\ 15 & 5 & 2 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n, \text{ or } L \text{ say,}$$

whose first column has for elements the so-called triangular numbers $(2)_2$, $(3)_2$, $(4)_2$, . . . , and whose s^{th} column, excluding the first, has for elements $s - 2$ zeros followed by $s - 1$ units, $s - 1$ twos, $s - 1$ threes and so on. In regard to the character of the generalization aimed at by Cipolla it must suffice to say that the new determinant has zeros where L has zeros, has $G(r)$

where L has $\frac{1}{2}r(r+1)$ and elsewhere has $F(p)$ where L has p, it being understood that F and G are suitably chosen arithmetical functions.

AIYAR, S. N. (1915[1]/4)

[Questions 630, 718. *Journ. Indian Math. Soc.*, viii. pp. 57–58: ix. pp. 85–87.]

The determinant involved here is the persymmetric recurrent first noted by Brioschi in 1858 (*Hist.*, iii. pp. 208–209, 214), the new result being that *if*

$$A_r = \frac{(a-b)(a-bx)\ \ldots\ (a-bx^{r-1})}{(1-x)(1-x^2)\ \ldots\ (1-x^r)}$$

and

$$B_r = \frac{(b-a)(b-ax)\ \ldots\ (b-ax^{r-1})}{(1-x)(1-x^2)\ \ldots\ (1-x^r)}$$

then the persymmetric recurrent of $A_1, A_2, \ldots, A_n$ *is equal to* $(-1)^n B_n$, *and the persymmetric recurrent of* $B_1, B_2, \ldots, B_n$ *is equal to* $(-1)^n A_n$. Proofs are given, but when we come to know that the A's and B's are such that

$$A_1 + B_1 = 0,$$
$$A_2 + A_1 B_1 + B_2 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

we are more fully satisfied (*Hist.*, iii. pp. 246–247).

SCHWATT, I. J. (1915/5, 10/8)

[On the expansion of a continued product. *Giornale di Mat.*, liii. pp. 186–189.]

[On the expansion of a continued trigonometric product. *Archiv d. Math. u. Phys.*, (3) xxiv. pp. 189–192.]

The two recurrents of these notes are found as equivalents for the coefficient of $x^k/k!$ in the expansion of $(1-x)(1-x^2)$ $\ldots (1-x^t)$. The second is the more interesting, its form dating back to 1859 (*Hist.*, iii. p. 209).

WHITE, C. E. ($1915/_{10}$)

[A general theorem regarding roots and solutions by determinants. *Tôhoku Math. Journ.*, viii. pp. 73–77.]

A curious determinant form for $a_0x^n + a_1x^{n-1} + \ldots + a_n$ is employed here, for example,

$$ax^4 + bx^3 + cx^2 + dx + e = \begin{vmatrix} ax+b & -1 & . & . \\ . & x & -1 & . \\ cx & . & x & -1 \\ dx+e & . & . & x \end{vmatrix};$$

and the mode of obtaining it is equally odd. It is desirable therefore to point out that we can deduce it from the already known recurrent

$$\begin{vmatrix} a & -1 & . & . & . \\ b & x & -1 & . & . \\ c & . & x & -1 & . \\ d & . & . & x & -1 \\ e & . & . & . & x \end{vmatrix}$$

by performing the operation $\text{row}_4 + \frac{1}{x}\,\text{row}_5$, transferring the factor x from the 5th column to the 1s and then performing $\text{col}_1 - b\,\text{col}_2$.

KENYON, A. M. ($1915/_{11}$): BELL, E. T. ($1918^1/_5$): SZÁSZ, O. ($1918^{17}/_{10}$)

[Question 443. *American Math. Monthly*, xxii. p. 308.]

[Some remarkable determinants of integers. *Bull. American Math. Soc.*, xxiv. pp. 376–380.]

[Determinantendarstellungen einiger Zahlen theoretischer Functionen. *Archiv d. Math. u. Phys.*, (2) xxvii. pp. 121–126.]

All the recurrents here considered originate in the study of the Theory of Integers. The first contribution is merely an isolated result, the recurrent being, strangely enough, identical in form with that derived by Brioschi in 1854 from Newton's relations between the coefficients of an equation and the sums of like powers of the roots (*Hist.*, ii. pp. 211–212). The second

contribution concerns determinants which, when of the 6^{th} order, are of the form

$$\begin{vmatrix} G(1) & F(1) & . & . & . & . \\ G(2) & F(2) & F(1) & . & . & . \\ G(3) & F(3) & F(1) & F(1) & . & . \\ G(4) & F(4) & F(2) & F(1) & F(1) & . \\ G(5) & F(5) & F(2) & F(1) & F(1) & F(1) \\ G(6) & F(6) & F(3) & F(2) & F(1) & F(1) \end{vmatrix},$$

where the k^{th} column consists of $k-2$ zeros and $k-1$ of each of the F's in order until n elements in all have been obtained. By the interchange of the F's and the G's the determinants occur in pairs: and a large number of special pairs are considered. The third contribution establishes a new reversion-theorem, and applies it to several of the more important functions peculiar to the Theory of Integers, the first column of the resulting recurrent being

$$G(2) - G(1),\ G(3) - G(1),\ \ldots,\ G(n) - G(1),$$

and the elements below the bordering diagonal of units being either units or zeros.

SMITH, E. R. (1916/3): WILLIAMS, K. P. (1916 1/2)

[Decomposition into partial fractions. *Math. Teacher*, viii. pp. 132–144.]

[Relating to some determinants connected with the Bernoulli numbers. *American Math. Monthly*, xxiii. pp. 263–264.]

The portion which concerns determinants in the first of these articles (pp. 139–144) is quite elementary.

The recurrents in the second article are got by eliminating $B_1, B_2, \ldots, B_n$ from n of one set of recurrent relations and one of another set: for example,

$$\begin{vmatrix} 1 & 3_2 & . & . & . \\ 3 & 5_2 & 5_4 & . & . \\ 5 & 7_2 & 7_4 & 7_6 & . \\ 7 & 9_2 & 9_4 & 9_6 & 9_8 \\ 8 & 10_2 & 10_4 & 10_6 & 10_8 \end{vmatrix} = 0.$$

WATSON, G. N. (1916)

[Some simple expansion-formulæ. *Messenger of Math.*, xlvi. pp. 97–101.]

We have already several times had occasion to refer to expansions of the type here dealt with, namely, the expansion of

$$(a_0 + a_1x + a_2x^2 + \dots)^\mu$$

in ascending powers of x, the more frequently occurring cases being that where $\mu = -1$ and that where $\mu = \frac{1}{2}$ (*Hist.*, ii. pp. 212–214, iv. p. 229). In regard to the former of these it had to be noted that two *different* expansions were available,—or, rather, let us say, two expansions whose undoubted identity was anything but obvious, being in fact bound up with the inferred identity of two recurrents (*Hist.*, iv. pp. 239–240). What we have now reached is in effect a generalization of this latter identity, the author proving that if

$$(a_0 + a_1x + a_2x^2 + \dots)^{-\mu} = g_0 + g_1x + g_2x^2 + \dots$$

then, in the first place, as Segar had shown in 1892 (*Hist.*, iv. pp. 236–237)

$$g_r = \frac{(-1)^r}{r!\,a_0{}^r}\begin{vmatrix} \mu a_1 & 1\cdot a_0 & \cdot & \dots & \cdot \\ 2\mu a_2 & (\mu+1)a_1 & 2a_0 & \dots & \cdot \\ 3\mu a_3 & (2\mu+1)a_2 & (\mu+2)a_1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r\mu a_r & (r\mu-\mu+1)a_{r-1} & (r\mu-2\mu+2)a_{r-2} & \dots & (\mu+r-1)a_1 \end{vmatrix}$$

where the coefficients of the a's in the first column form an arithmetical progression with common difference μ, and the coefficients of the a's in every row form an arithmetical progression with common difference $-(m-1)$; and in the second place

$$g_r = \frac{(-1)^r}{r!\,a_0{}^r}\begin{vmatrix} (\mu+1)a_1 & 1\cdot a_0 & \cdot & \dots & \cdot \\ (2\mu+2)a_2 & (\mu+2)a_1 & 2a_0 & \dots & \cdot \\ (3\mu+3)a_3 & (2\mu+3)a_2 & (\mu+3)a_1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r\mu a_r & (r-1)\mu a_{r-1} & (r-2)\mu a_{r-2} & \dots & \mu a_1 \end{vmatrix}$$

where in the first column the coefficients of the a's save the coefficient of a_r form an arithmetical progression with common difference $\mu + 1$, and the coefficients of the a's in every row form an arithmetical progression with common difference $-\mu$. Unfortunately the question of the possible transformation of the one determinant into the other does not engage the author's attention.

SANJANA, K. J. (1917/8)

[Question 883. *Journ. Indian Math. Soc.*, ix. p. 170; xii. pp. 105–106.]

The n-line recurrent here has for its r^{th} row

$$\frac{1}{(2r-1)!},\ \frac{1}{(2r-2)!},\ \frac{1}{(2r-3)!},\ \ldots,$$

and for its value

$$\left\{1^{2n}\cdot 3^{2n-1}\cdot 5^{2n-2}\ldots(4n-1)^1\right\}^{-1}.$$

MUIR, T. (1918¹/3)

[Egalité de deux déterminants. *L'Intermédiaire des Math.*, xxv pp. 33–34.]

The equality in question is that which we have noted above as being due to Ono, who set it for proof in 1913. The case for the 5^{th} order is

$$\begin{vmatrix} 2a & 1 & . & . & . \\ 4b & 3a & 2 & . & . \\ 6c & 5b & 4a & 3 & . \\ 8d & 7c & 6b & 5a & 4 \\ 5e & 4d & 3c & 2b & a \end{vmatrix} = 5!\begin{vmatrix} a & 1 & . & . & . \\ b & a & 1 & . & . \\ c & b & a & 1 & . \\ d & c & b & a & 1 \\ e & d & c & b & a \end{vmatrix},$$

where, according to the law of the preceding rows, we should expect the 5^{th} row on the left to be $10e$, $9d$, $8c$, $7b$, $6a$. The operations

$$\text{row}_5 + a\ \text{row}_4 + b\ \text{row}_3 + c\ \text{row}_2 + d\ \text{row}_1,$$
$$\text{row}_4 + a\ \text{row}_3 + b\ \text{row}_2 + c\ \text{row}_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

performed on the first determinant enable us to remove from it the factor 5! leaving the cofactor

$$\begin{vmatrix} 2a & 1 & \cdot & \cdot & \cdot \\ 2b+a^2 & 2a & 1 & & \cdot \\ 2c+2ba & 2b+a^2 & 2a & 1 & \cdot \\ 2d+2ca+b^2 & 2c+2ba & 2b+a^2 & 2a & 1 \\ e+2da+2cb & d+2ca+b^2 & c+2ba & b+a^2 & a \end{vmatrix};$$

then by performing on this the operations

$$\begin{gathered} \mathrm{col}_4 - a\,\mathrm{col}_5 \\ \mathrm{col}_3 - a(\text{new } \mathrm{col}_4) - b\,\mathrm{col}_5, \\ \cdots\cdots\cdots\cdots \end{gathered}$$

we obtain the result desired.

MUIR, T. (1918$^1/_2$, $/_8$)

[Questions 18592, 18698. *Math. Quest. and Sol.*, v. Pt. 2; vi. pp. 76–77.]

Two other curious instances of the equivalence of two recurrents, namely,

$$\begin{vmatrix} a & 1 & \cdot & \cdot \\ mb & (m+1)a & 1 & \cdot \\ (m^2-1)c & (m^2+m-1)b & ma & 1 \\ m(m+1)d & (m+1)^2c & (m+1)b & a \end{vmatrix} = m(m+1)\begin{vmatrix} a & 1 & \cdot & \cdot \\ b & a & 1 & \cdot \\ c & b & a & 1 \\ d & c & b & a \end{vmatrix},$$

$$\begin{vmatrix} 1 & -2 & \cdot & \cdot & \cdot \\ 3 & 1 & -4 & \cdot & \cdot \\ 10 & 3 & 1 & -6 & \cdot \\ 35 & 10 & 3 & 1 & -8 \\ 14 & 5 & 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & \cdot & \cdot & \cdot \\ 3 & 1 & 4 & \cdot & \cdot \\ 10 & 3 & 1 & 6 & \cdot \\ 35 & 10 & 3 & 1 & 8 \\ 42 & 14 & 5 & 2 & 1 \end{vmatrix},$$

where the integers 1, 1, 2, 5, 14, 42, . . . of the last two recurrents belong to the type

$$(2k+1)_k \div (2k+1).$$

WHITTAKER, E. T. (1918 8/8)

[A formula for the solution of algebraic or transcendental equations. *Proceed. Edinburgh Math. Soc.*, xxxvi. pp. 103–106.]

The theorem established here is that the numerically smallest root of the equation

$$0 = a_0 + a_1x + a_2x^2 + \ldots$$

is given by the series

$$-\frac{a_0}{a_1} - \frac{a_0{}^2 a_2}{a_1 \begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}} - \frac{a_0{}^3 \begin{vmatrix} a_2 & a_1 \\ a_3 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_0 & \cdot \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix}} \quad \ldots$$

The first point of interest is connected with the summing of the series, the sum of r terms being shown to be

$$-a_0 \begin{vmatrix} a_1 & a_0 & \cdot & \cdot & \ldots \\ a_2 & a_1 & a_0 & \cdot & \ldots \\ a_3 & a_2 & a_1 & a_0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_{r-1} \div \begin{vmatrix} a_1 & a_0 & \cdot & \cdot & \ldots \\ a_2 & a_1 & a_0 & \cdot & \ldots \\ a_3 & a_2 & a_1 & a_0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_r;$$

and a second point is that this quotient is equal to the result of dividing the coefficient of x^{r-1} by the coefficient of x^r in the expansion of

$$(a_0 + a_1x + a_2x^2 + \ldots)^{-1}.$$

These bring us into touch with previous related work (*Hist.*, ii. p. 40; iii. pp. 423–426).

MUIR, T. (1918 14/8)

[Note on recurrents resolvable into a sequence of odd integers. *Transac. R. Soc. S. Africa*, viii. pp. 27–32.]

The recurrents here discussed are, save for their last rows, of the forms

$$\begin{vmatrix} 1 & 2 & . & . & \dots \\ (3)_1 & 1 & 4 & . & \dots \\ (5)_2 & (3)_1 & 1 & 6 & \dots \\ (7)_3 & (5)_2 & (3)_1 & 1 & \dots \\ . & . & . & . & . \end{vmatrix}_n, \quad \begin{vmatrix} 1 & -2 & . & . & \dots \\ (3)_1 & 1 & -4 & . & \dots \\ (5)_2 & (3)_1 & 1 & -6 & \dots \\ (7)_3 & (5)_2 & (3)_1 & 1 & \dots \\ . & . & . & . & . \end{vmatrix}_n.$$

Calling the former $S_n(m)$ and the latter $R_n(m)$ when the last row in both is

$$(2n+m-4)_{n-1},\quad (2n+m-6)_{n-2},\quad \dots,\quad (m+2)_2,\quad m_1,\quad 1$$

we have

$$\begin{aligned} S_n(m) &= -(2n+2m-5)S_{n-1}(m+2) \\ &= (-1)^{n-1}(2n+2m-5)(2n+2m-3)\dots(4n+2m-9), \end{aligned}$$

and $R_n(m) = (2n+2m-3)(2n+2m-1)\dots(4n+2m-7)$. If m be taken equal to 3 these give Reich's results of 1892 (*Hist.*, iv. p. 237). A number of other related equalities are obtained, the last being

$$\begin{vmatrix} 1 & 2 & . & . & . \\ 3 & 1 & 4 & . & . \\ 10 & 3 & 1 & 6 & . \\ 35 & 10 & 3 & 1 & 8 \\ (m+7)_4 & (m+5)_3 & (m+3)_2 & (m+1)_1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & . & . & . \\ 3 & 1 & -4 & . & . \\ 10 & 3 & 1 & -6 & . \\ 35 & 10 & 3 & 1 & -8 \\ (m+6)_4 & (m+4)_3 & (m+2)_2 & m_1 & 1 \end{vmatrix}.$$

BOUCHARY, J. (1918$^1/_{11}$): SCHMIDT, ÉD. (1919/$_6$)

[Question 2379. *Nouv. Annales de Math.*, (4) xviii. p. 439.]
[Essais arithmétiques. *Mém. . . . Soc. . . . Sci.* (Liège), (3) xi. iv + 71 pp.]

The first of these deals with an aggregate of recurrents, of which the separate individuals are known (*Hist.*, iv. p. 231).

The ' essais ', so far as our subject is concerned, are mainly applicational. The author uses the word ' recurrent ' in a peculiar way.

NEVILLE, E. H. (1919[1]/11)

[Two trigonometrical determinants. *Messenger of Math.*, xlix. pp. 107–111.]

The determinants in question are recurrents given as equivalents for $2\cos nx$ and $\sin nx/\sin x$. The originating series of the first is

$$(2\cos x)^n = 2\cos nx + 2n_1\cos(n-2)x + 2n_2\cos(n-4)x + \ldots$$

and the derived result

$$2\cos nx = \begin{vmatrix} (2\cos x)^n & n_1 & n_2 & n_3 & \ldots \\ (2\cos x)^{n-2} & 1 & (n-2)_1 & (n-2)_2 & \ldots \\ (2\cos x)^{n-4} & . & 1 & (n-4)_1 & \ldots \\ (2\cos x)^{n-6} & . & . & 1 & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

From this there is deduced the ordinary non-determinant form of which the following variant may not be inappropriate:

$$\begin{aligned} \cos nx = {} & \cos^n x\,(1 + n_2 + n_4 + n_6 + \ldots) \\ & + \cos^{n-2}x\,(-n_2 - 2n_4 - 3n_6 - \ldots) \\ & + \cos^{n-4}x\,(n_4 + 3n_6 + \ldots) \\ & + \cos^{n-6}x\,(-n_6 - \ldots) \\ & \cdots\cdots\cdots\cdots \end{aligned}$$

CHAPTER IX

Wronskians, from 1891 to 1919

The interest awakened in the Wronskian during the period 1880–1899 is more than maintained in the next twenty-year period, the number of writings occupied with it being greater by a half. Many of them, however, circle round the test for linear independence, and thus only indirectly and sparingly contribute to our knowledge of the properties of the determinant.

HEYMANN, W. (1891)

[Zur Theorie der Differenzengleichungen. *Crelle's Journ.*, cix. pp. 112–117.]

This paper deserves attention in passing because it furnishes the following analogue to a theorem concerning the Wronskian and differen*tial*-equations: *In a linear difference-equation*

$$y_{x+n} + p_{n-1}y_{x+n-1} + \cdots + p_0 y_x = 0$$

the determinant of a fundamental system of solutions

$$\begin{vmatrix} y_x^{(1)} & y_x^{(2)} & \dots & y_x^{(n)} \\ y_{x+1}^{(1)} & y_{x+1}^{(2)} & \dots & y_{x+1}^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ y_{x+n-1}^{(1)} & y_{x+n-1}^{(2)} & \dots & y_{x+n-1}^{(n)} \end{vmatrix}, \text{ or } \mathrm{D}_x \text{ say,}$$

satisfies the homogeneous linear difference-equation of the first order

$$D_{x+1} = (-1)^n p_0 D_x.$$

IGEL, B. (1893)

[Ueber eine Determinantenbeziehung in der Theorie der Differentialgleichungen. *Monatshefte f. Math. u. Phys.*, iv. pp. 380–394].

The title here is the same as that of Königsberger's paper of 1889 (*Hist.*, iv. p. 246), and the contents are a detailed critical examination of the contents of that paper. The subject, however, of the successive differentiation of a Wronskian, which Königsberger had used as an auxiliary in his work, is not thereby directly affected.

The connection of Wronskians with differential-equations, to which Frobenius drew attention in 1873 (*Hist.*, iii. pp. 250–254), may be further studied in the papers of Cels, Grünfeld, and others noted by us in the list of auxiliary writings at the end of this chapter. Of course standard textbooks on differential equations, like those of Forsyth, Heffter, Schlesinger, Craig, . . ., will also be found useful.

BÖTTCHER, L. E. ($1900^{18}/_6$)

[Owlasnościach pewnych wyznaczników funkcyjnych. *Rozprawy Akad.* . . . (Krakow), xxxviii. pp. 382–389: or *Bull. Internat.* . . . *Acad. des Sci.*, 1900, pp. 227–228.]

The determinant here investigated is Grévy's of 1894 (*Hist.*, iv. p. 248), namely, the determinant which has $F_r(x_{s-1})$ for its $(r, s)^{\text{th}}$ element and in which $x_1 = f(x)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, . . . The paper is merely noted in passing because of the analogy between the determinant and the Wronskian, the first column consisting of n functions of x and every other column being got from the column immediately preceding by the performance of a fixed operation, thus leaving the determinant itself a function of only one variable (x).

BOECHER, M. ($1900^{15}/_{9}$, $/_{11}$)

[The linear independence of functions of one variable. *Bull. American Math. Soc.*, (2) vii. pp. 120–121.]

[The theory of linear dependence. *Annals of Math.*, (2) ii. pp. 81–96.]

From a thoughtful examination of the Wronskian test for linear dependence the author apparently was led to the consideration of Linear Dependence in general: and fortunately he decided to put the result of his study into the form of an expository article. The first section of this is occupied with definitions and general theorems, the second with the linear dependence of sets of quantities, the third with that of polynomials, and the fourth, for our special benefit, with that of functions of a single variable. The outcome of this last discussion, extending to five pages (pp. 89–94), is the formulation of the Wronskian test in two portions, namely: (1) *The identical vanishing of the Wronskian of* n *analytic functions is a necessary and sufficient condition for their linear dependence;* (2) *The identical vanishing of the Wronskian of* n *functions of a real variable, each of which has at every point of a certain interval finite derivatives of the first* n — 1 *orders, is a sufficient condition for linear dependence, provided that* n — 1 *of the functions can be so selected that their Wronskian and its first derivative do not vanish together at any point of the interval.* The second of these, of course, enters into competition with the corresponding formal statements of Peano and Vivanti (*Hist.*, iv. pp. 249–250). A useful section is added under the heading ‘ applications ’, in which the results quoted are left to be proved by the reader.

BOECHER, M. ($1900/_{12}$, $1901^{20}/_{7}$)

[Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence. *Transac. American Math. Soc.*, ii. pp. 139–149.]

[The Wronskians of functions of a real variable. *Bull. American Math. Soc.*, (2) viii. pp. 53–63.]

There is more in the first of these papers than the title would

imply, the last two pages being devoted to the establishment of a pure theorem on Wronskians, namely: *If* $y_1, y_2, \ldots, y_{n+1}$ *be functions of* x *which at every point of an interval* I *have continuous derivates of the first* n *orders, and if the Wronskian of* $y_1, y_2, \ldots y_n$ *vanishes identically, then the Wronskian of* $y_1, y_2, \ldots, y_n, y_{n+1}$ *likewise vanishes identically.* As a preliminary to it proof is given of the following lemma, the original form of which is due to Pasch (*Hist.*, iii. p. 255): *If* $y_1, y_2, \ldots, y_n$ *be functions of* x *which at every point of* I *have finite derivates of the first* k *orders* ($k \geqq n$), *and if their Wronskian vanishes identically, then, except at points where* $W(y_1, y_2, \ldots, y_{n-1}) = 0$, *all the* n-*line minors of the array*

$$\left\| \begin{array}{cccc} y_1 & y_1' & \ldots & y_1^{(k)} \\ y_2 & y_2' & \ldots & y_2^{(k)} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ y_n & y_n' & \ldots & y_n^{(k)} \end{array} \right\| \quad \textit{are zero.}$$

GULDBERG, A. (1903 12/10)

[Sur les équations linéaires aux différences finies. *Comptes rendus . . . Acad. des Sci.* (Paris), cxxxvii. pp. 560-562, 614–615.]

A paragraph of this (§ 3, p. 561) has the same interest for us as Heymann's paper of 1891, namely, in that it provides an analogue to a theorem in differential-equations in which the Wronskian figures. In regard to this contribution it is enough to say that it is a simple deduction from the fact that the difference-equation

$$y_{x+n} + p_{n-1}\, y_{x+n-1} + \ldots + p_0\, y_x = 0$$

can be expressed by means of a fundamental system of solutions $y_x^{(1)}, y_x^{(2)}, \ldots, y_x^{(n)}$ in the form

$$\left| \begin{array}{cccc} y_x & y_{x+1} & \ldots & y_{x+n} \\ y_x^{(1)} & y_{x+1}^{(1)} & \ldots & y_{x+n}^{(1)} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ y_x^{(n)} & y_{x+1}^{(n)} & \ldots & y_{x+n}^{(n)} \end{array} \right| = 0.$$

MINA, L. (1904$^{27}/_{9}$)

[Formole generali delle derivate successive d'una funzione espresse mediante quella della sua inversa. *Giornale di Mat.*, xliii. pp. 196–212.]

The formulæ in question, three in number, are expressions for $d^n x/dy^n$ in terms of the differential coefficients of y with respect to x: and the paper calls for our attention because in one of the formulæ a Wronskian figures as an essential, the equality in fact being

$$\frac{d^n x}{dy^n} = \frac{W\left(\frac{d^2 y}{dx^2}, \frac{d^2 y^2}{dx^2}, \ldots, \frac{d^2 y^{n-1}}{dx^2}\right)}{1!\ 2! \ldots (n-1)!} \div \left(\frac{dy}{dx}\right)^{\frac{1}{2}n(n-1)}$$

As is not unnatural the mode of establishing it is indirect. The first step consists in showing that if $F_0, F_1, F_2, \ldots, F_n$ be functions of y and y be a function of x, then the relation between the two Wronskians of the F's is

$$W_y \cdot \left(\frac{dy}{dx}\right)^{\frac{1}{2}n(n+1)} = W_x.$$

Next the special case of this is taken where

$$F_0, F_1, F_2, \ldots, F_{n-1}, F_n = 1, y, \frac{y^2}{2}, \ldots, \frac{y^{n-1}}{(n-1)!}, x;$$

and it is seen with ease that then

$$W_y = \frac{d^n x}{dy^n},$$

while after some transformation and reduction it is shown that

$$W_x = (-1)^n \frac{W\left(\frac{d^2 y}{dx^2}, \frac{d^2 y^2}{dx^2}, \ldots, \frac{d^2 y^{n-1}}{dx^2}\right)}{1!\ 2! \ldots (n-1)!}.$$

Substitution then at once gives the desired result.

OCCHIPINTI, R. (1904/$_{11-12}$)

[Su alcuni determinanti di funzioni composte. *Periodico di Mat.*, (3) ii. pp. 132–134.]

The functions referred to are $yy_1, yy_2, \ldots, yy_n$, and the determinants are the Wronskian and Jacobian. In contents the paper resembles Casorati's of 1874 (*Hist.*, iii. pp. 254–255), but is not so thorough; in the case of the Wronskian a still earlier predecessor has to be referred to, namely, Christoffel (*Hist.*, ii. p. 227).

MLODZĚJEVSKIJ, B. K. (1905)

[On a generalization of Wronksi's determinant (In Russian). *Mat. Sbornik* (Moscow), xxv. pp. 474–477.]

The determinant in question is identical with the n-line Wronskian in its first h rows, but has constants in the remaining places.*

MEDER, A. (1906/$_1$)

[Ueber die Determinante von Wronski. *Monatshefte f. Math. u. Phys.*, xvii. pp. 19–43.]

The writer supplements his title by saying that his subject is those minors of the Wronskian $W(y_1, y_2, \ldots)$ whose elements are taken from any a columns and from the first a rows. As a matter of fact his subject is a theorem in which the data are the vanishing, for a particular value of the variable, of $W(y_l)$, $W(y_l, y_m)$, $W(y_l, y_m, y_n), \ldots$, and their successive differential-coefficients up to but not including the $\nu_1^{\text{th}}, \nu_2^{\text{th}}, \nu_3^{\text{th}}, \ldots$, respectively. The proof of this theorem occupies eighteen pages (pp. 20–37), and the enunciation of it the whole of the 38th page, the first and main part of the latter being that

$$\nu_2 \geq 2\nu_1, \quad \nu_3 \geq 2\nu_2 - \nu_1, \quad \nu_4 \geq 2\nu_3 - \nu_2, \quad \ldots$$

In the remaining five pages are considered what, if any, modifications in the theorem are necessary when certain inequalities do not hold for all values of $l, m, \ldots$.

* Unfortunately we have not seen the paper.

PASCAL, E. (1906/$_5$)

[Sopra una proprietà dei determinanti Wronskiani. *Atti . . . Accad. delle Sci.* (Torino), xli. pp. 1081–1083.]

Here once more the subject is the Wronskian test for linear dependence. The contribution made, however, is quite fresh, being to the effect that the condition $W = 0$ is equivalent to the condition

$$y_1 \int W_{(1)}\,dx - y_2 \int W_{(2)}\,dx + \ldots + (-1)^{n-1}\, y_n \int W_{(n)}\,dx = 0,$$

where W_r stands for the Wronskian of all the y's except y_r. The proof that the latter entails the former is simpler than that of the converse.

CURTISS, D. R. (1906/$_5$, 1908^{13}/$_2$)

[On certain properties of Wronskians and related matrices. *Bull. American Math. Soc.*, xii. pp. 482–485.]

[The vanishing of the Wronskian and the problem of linear dependence. *Math. Annalen*, lxv. pp. 282–298.]

The first of these papers contains only the formal statement of five theorems with a promise of proofs to be published later—a brief anticipation, in fact, of something resembling the second paper, the plan of which doubtless grew as the writer continued his study. The latter is not, as one might possibly expect, a continuation of Boecher's work, though manifestly influenced thereby, but a thorough-going reinvestigation of the whole subject with non-analytic functions in view from the outset. The main part of the first section is occupied with the n-line determinants of the n-by-$(k + 1)$ Wronskian array: in the second section the test itself is dealt with, its newest form being: *If* $y_1, y_2, \ldots, y_n$ *be functions of* x *which at every point of the interval* I *have finite derivates of the first* k *orders* ($k \geq n - 1$) *while the* $(n - 1)$*-line determinants of the* $(n - 1)$*-by-*$(k - 1)$ *Wronskian array do not all vanish at any point of* I, *and if* $W(y_1, y_2, \ldots, y_n)$ *be identically zero, then* $y_1, y_2, \ldots, y_n$ *are linearly dependent.* The third section shows the usefulness of taking into consideration the rank (non-zero) of the n-by-$(k + 1)$ Wronskian array: and the fourth returns

to the test for the purpose of proving its newest form to be essentially identical with one of Boecher's.

ORLANDO, L. ($1908^6/_6$): PASTA, G. (1908)

[Sul determinante di Wronski. *Atti . . . Accad. dei Lincei* (*Rendic.*), (5) xvii. pp. 717–720.]

[Sul dipendenza lineare di *n* funzioni ad *n* variabili. *Rivista di Fis., Mat. e Sci. Nat.*, ix. pp. 71–73.]

The only matter brought forward by Orlando is the Wronskian test and especially the form of it proposed by Peano (*Hist.*, iv. pp. 249–250). The work done on the same subject by American writers is not referred to.

In the other paper the main subject is in connection with the corresponding part of Pasch's paper of 1874 (*Hist.*, iii. p. 155).

SIBIRANI, F. (1909)

[Sui sistemi di integrali independenti di *m* equazioni differenziali lineari. *Giornale di Mat.*, xlvii. pp. 156–163.]

In reality the problem tackled here is the finding of the Wronskian of m sets of n functions, each set of functions being the n linearly independent integrals of a differential equation

$$\frac{d^n y}{dx^n} = b_{k,n}\frac{d^{n-1}y}{dx^{n-1}} + b_{k,n-1}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + b_{k,1}y \quad (k = 1, \ldots, m).$$

In its first form of expression the said Wronskian is a determinant of the $(mn)^{\text{th}}$ order. It is readily found, however, to be changeable into the product of two determinants of that order, one of which is seen to be the product of the m Wronskians of the m sets of functions. Since none of these subordinate Wronskians can possibly be zero, they may be dispensed with, leaving the non-vanishing of their cofactor as the condition for the linear independence of the mn functions. By way of illustration the case where the b's of the differential equations are constants is dis-

cussed: also the case of this where the differential equations are of the form

$$\frac{d^n y}{dx^n} = b_k y:$$

and finally the further specialized case where the functions are

$$\sin x,\ \cos x,\ \sin 2x,\ \cos 2x,\ \ldots,\ \sin mx,\ \cos mx.$$

In the last case the Wronskian is found to be

$$(-1)^{m-1} \left| 1^0 2^2 3^4 \ldots m^{2m-2} \right| \quad \text{i.e.} \quad (-1)^{m-1} \frac{(2m)!!}{2^m (m!)^2}.$$

KOWALEWSKI, G. (1909)

[Einfürhrung in die Determinantentheorie. . . . Chapter XV, pp. 320–337.]

This chapter, which is headed 'Wronskian and Gramian determinants', contains six sections, of which only one is given to the Wronskian and all of that to the discussion of the test. Indeed the subject of the chapter is really Linear Dependence of Functions, and its value lies in the introduction and treatment of the other determinant

$$\left| \int_a^b y_r y_s dx \right|_n,$$

to which for no very cogent reason the name of Gram is attached.*

WALLENBERG, G. (1909 24/11)

[Beiträge zur Theorie der linearen Differenzengleichungen. *Sitzungsb. d. Berliner Math. Ges.*, ix. pp. 2–8.]

Here again we have a paper with a paragraph bearing on analogues in difference-equations to theorems involving the Wronskian in differential-equations. This time the theorem that is provided with a companion is that given by Fuchs in 1865 which expresses the determinant of a fundamental system of solutions as a product of integrals.

* See under Kowalewski in Chap. XI.

MUIR, T. ($1910^{13}/_6$)

[The theory of Wronskians in the historical order of development up to 1860. *Proceed. R. Soc. Edinburgh*, xxxi. pp. 296–303.]

This, our first separate paper on the history of Wronskians, gives short accounts of ten writings belonging to the period 1838–1858, the contributions of Wronski himself (1812–1817) and Schweins (1825) having been already dealt with elsewhere. Perhaps the most important of the ten is Christoffel's of 1857, in which compound Wronskians first made their appearance.

WESTENDORP, J. J. C. (1910)

[Over het differentieëren van determinanten, waarvan de elementen functies van één veranderlijke zijn. *Wiskundig Tijdschrift*, vii. pp. 131–133.]

The determinant referred to is the Wronskian, and what is given is the usual textbook proof * regarding its differential-coefficient.

MEDER, A. (1910)

[Ueber den Zusammenhang zwischen den Determinanten von Gram und Wronski. *Monatshefte f. Math. u. Phys.*, xxi. pp. 336–343.]

The connection in question is all the more interesting because of its unexpected character. The so-called Gram's determinant as usually written

$$\left| \int_a^b f_r f_s \, dx \right|_n$$

is a function of a and b, and the Wronskian of the same f's is a function of x; they may therefore be appropriately denoted by

$$G(a, b) \quad \text{and} \quad W(x)$$

respectively. Now Meder's connecting theorem is that *if we differentiate* G n^2 *times with respect to* b, *and in the result put*

* E.g. Muir's of 1882, § 195, p. 224.

b = a, *we shall obtain an arithmetical multiple of* W(a): or, in symbols,

$$\frac{d^{nn}G(a, b)}{db^{nn}} = N \cdot W(a)$$

where

$$N = \frac{n^2!}{1^1 \cdot 2^2 \ldots (n-1)^{n-1} n^n (n+1)^{n-1} \ldots (2n-1)^1}.$$

The case where $f_r = x^{r-1}$, which is used for the determination of N would serve better as an illustrative example were it not that W is then a constant.

CURTISS, D. R. (1911)

[Relations between the Gramian, the Wronskian, and a third determinant connected with the problem of linear dependence. *Bull. American Math. Soc.*, (2) xvii. pp. 462–467.]

The third determinant referred to, whose connection with the Wronskian we have to take note of, is the general alternant

$$|\, f_1(x_1) f_2(x_2) \ldots f_n(x_n) \,|, \text{ or } \mathfrak{A} \text{ say.}$$

If we differentiate it once with respect to x_2, then the result twice with respect to x_3, and so on, we obtain

$$\begin{vmatrix} f_1(x_1) & f_1'(x_2) & f_1''(x_3) & \ldots \\ f_2(x_1) & f_2'(x_2) & f_2''(x_3) & \ldots \\ f_3(x_1) & f_3'(x_2) & f_3''(x_3) & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n$$

which, on putting $x_1, x_2, \ldots$ each equal to x, manifestly becomes $W(x)$. This is the connection in question. A similar tie joining the Wronskian and Gram's determinant, but different from what Meder disclosed, is also sought for and found, the actual expression of it, however, not being ventured on. The remaining formula of the paper, although it does not involve W, we may give in passing, namely,

$$G = \frac{1}{n!} \int_a^b \ldots \int_a^b \mathfrak{A}^2 dx_1 \ldots dx_n.$$

It will be seen to be the case of Andréieff's equality of 1883 in which his two alternants are the same (*Hist.*, iv. p. 464).

VALIRON, G. (1911)

[Note sur les déterminants de Wronski. *Nouv. Annales de Math.*, (4) xi. pp. 151–153.]

Using the familiar theorem regarding a minor of the adjugate the author, on the supposition that $W(y_3, \ldots, y_n) \neq 0$, proves that

$$W(y_1, y_2, \ldots, y_n) = \frac{\{W(y_1, y_3, \ldots, y_n)\}^2}{W(y_3, \ldots, y_n)} \cdot \frac{d}{dx} \frac{W(y_2, \ldots, y_n)}{W(y_1, y_3, \ldots, y_n)};$$

and then on the further supposition that

$$W(y_1, y_3, \ldots, y_n) \neq 0$$

arrives at the linear dependence of the y's when

$$W(y_1, y_2, \ldots y_n) = 0.$$

TOLEDO, L. O. DE (1911/9)

[Propriedades del Wronskiano. *Revista . . . Soc. Mat. Española*, i. pp. 80–87.]

The author disclaims the bringing forward of any new properties, his object apparently being to provide a useful introduction to the theory such as should be found in every good textbook of determinants. In this he is fully successful, the matter being well chosen and the exposition clear. As an instance of a field for applications choice is made of the theory of linear differential equations.

HOWLAND, L. A. (1911)

[On the derivative of the quotient of two Wronskians. *American Math. Monthly*, xviii. pp. 219–221.]

This concerns the theorem which Frobenius had used in 1873 (*Hist.*, iii. pp. 250–254) in connection with differential equations, and the main point which the author makes is that "we can reverse the application and use the theory of linear differential equations to establish the formula."

KELLOGG, O. D. ($1912^{18}/_{12}$)

[Sur l'indépendance linéaire des fonctions de plusieurs variables. *Bull. Soc. Math. de France*, xli. pp. 19–21 of the 'Comptes rendus des séances'.]

The Wronskian test for linear dependence naturally suggests inquiry as to an analogue in the case of several variables: hence Pasch's proposal of 1874 (*Hist.*, iii. pp. 255–256), Pasta's of 1908 (above), and that to which we have now come. Here the utilized array of differential-coefficients is, when the variables are two in number,

$$\left\| \begin{matrix} \phi_1 & \dfrac{\partial\phi_1}{\partial x} & \dfrac{\partial\phi_1}{\partial y} & \dfrac{\partial^2\phi_1}{\partial x^2} & \dfrac{\partial^2\phi_1}{\partial x\partial y} & \cdots & \dfrac{\partial^n\phi_1}{\partial y^n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|$$

so that, if the number of functions be n the number of columns is $\frac{1}{2}n(n+1)$.

MARTINOTTI, P. ($1913^{30}/_1$, $^9/_2$)

[Sul Wronskiano. *Rendic. del R. Istituto Lombardo* . . ., (2) xlvi. pp. 133–136.]

[Il Wronskiano e la dipendenza lineare di n funzioni di una variabile reale. *Rendic. del Circolo Mat.* (Palermo), xxxv. pp. 384–393.]

The first paper here is evidently the result of a critical study of Vivanti's of 1898 (*Hist.*, iv. pp. 249–250), made, however, without taking into consideration the series of other papers dealing with the same well-worn subject—Peano's views on the dependence-test.

The second paper, however, has a broader basis, the contributions of Boecher ($1900/_{12}$), Mlodzějevskij (1904), Orlando (1908) being now referred to in the introductory section. In other respects also it is better and fuller. It must suffice, however, to give only a case of the general theorem reached, namely, *When the functions concerned are developable in Taylor series, the vanishing of the Wronskian is a necessary and sufficient condition for them being connected by a homogeneous linear relation with non-zero coefficients.*

HAYASHI, T. (1914/6)

[A determinantal theorem. *Tôhoku Math. Journ.*, v. pp. 202–203.]

The theorem in question is the simplest case of the factorization of a compound Wronskian, the case to which special attention was given by Frobenius in 1873 (*Hist.*, iii. p. 252), but which had already been put on record by Christoffel in 1857 (*Hist.*, ii. p. 227). It might have been well had the author noted that the example which he gives to illustrate his theorem is true not merely for Wronskians but for all determinants whatever (Muir's textbook § 197, p. 225).

SELLERIO, A. (1915[20]/2)

[Su una particolare equazione differenziale. *Rendic. del Circolo Mat.* (Palermo), xl. pp. 27–28.]

The equation in question is that whose statement involves a persymmetric Wronskian: and the main result given is that *the equation*

$$W(y, y', y'', \ldots, y^{(n-1)}) = pe^{qx}$$

is equivalent to

$$a_0y + a_1y' + \ldots + a_{n-2}y^{(n-2)} - qy^{(n-1)} + y^{(n)} = 0$$

where p, q *and the* a's *are constants.* Reference might have been made to Sylvester's paper of 1862 (*Hist.*, iii. pp. 316–317).

SUTÔ, O. (1915/6)

[On the vanishing of certain determinants. *Tôhoku Math. Journ.*, vii. pp. 38–53.]

This contribution, ostensibly on determinants, evidently owes its origin to a study of the subject of Linear Dependence of Functions, for about two-thirds of it is directly concerned therewith. The five pages of introduction, on the other hand, really deal with determinants, the particular matter sought to be cleared up being the evanescence of arrays. As a preliminary the author makes an addition to the indexing of non-nullity. The auxiliary concept appears in the following definition: *If in a rectangular array there be* p *rows whose* p*-line determinants all vanish, and*

also somewhere in the array a (p — 1)*-line determinant that does not vanish,* p *is called the row-index* (of non-nullity) *of the array.* Of course in a square array there is a similarly definable column-index: so that we have now in all three indexes available for use in the study of a vanishing determinant. For example, in the determinant

$$\begin{vmatrix} x & 2x & x^2 & x^3 & x^2+x^3 \\ 1 & 2 & 2y & 3y^2 & 2y+3y^2 \\ . & . & 2 & 6z & 2+6z \\ . & . & . & 6 & 6 \\ . & . & . & . & . \end{vmatrix}$$

the row-index is 1, the column-index 2, and the rank 3. Grades, and therefore indexes, of linear dependence also are recognized; for we are told that if in an m-by-n array there exist q rows each of which is linearly dependent on the remaining $m-q$ rows, then the m rows are said to be 'linearly dependent of order q'.* This being premised a fundamental theorem on evanescence is formulated and proved, namely, *In order that all the* n*-line minors of the array*

$$\begin{Vmatrix} \phi_{11}(x_1) & \phi_{12}(x_2) & \dots & \phi_{1n}(x_n) \\ \phi_{21}(x_1) & \phi_{22}(x_2) & \dots & \phi_{2n}(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{m1}(x_1) & \phi_{m2}(x_2) & \dots & \phi_{mn}(x_n) \end{Vmatrix} \quad (m \geqq n)$$

may vanish, it is necessary and sufficient that the rows, with respect to some number s *of the variables, have a linear dependence of order* m — s + 1. The proof is gradational.

The remaining portion of the paper, if read along with Curtiss' paper of 1911 and other related papers mentioned, will be found helpful.

BOECHER, M. (1916/1)

[On the Wronskian test for linear dependence. *Annals of Math.*, xvii. pp. 167–168.]

After an experience of fifteen years Boecher makes known his

* The expression seems hardly commendable: perhaps it would be better to say 'have a linear dependence of order q' or 'have a q-fold linear dependence'.

creed and practice as a teacher in regard to the Wronskian's bearing on linear dependence. His creed is embraced in a pair of theorems (1) *If* $y_1(x), \ldots, y_n(x)$ *are throughout the interval* (a, b) *analytic functions whose Wronskian vanishes identically, then* $y_1, \ldots, y_n$ *are linearly dependent throughout* (a, b): (2) *If* $y_1, \ldots, y_n$ *satisfy the homogeneous linear differential equation*

$$\frac{d^k y}{dx^k} + p_1\frac{d^{k-1}y}{dx^{k-1}} + \ldots + p_k y = 0$$

at every point of an interval (a, b) *throughout which* $p_1, \ldots, p_k$ *are continuous, and if* $W(y_1, \ldots, y_n) \equiv 0$ *in* (a, b), *then* $y_1, \ldots, y_n$ *are linearly dependent in* (a, b). His practice is to begin by establishing the basic lemma, *If* $y_1, \ldots, y_n$ *have at each point of* (a, b) *finite derivatives of the first* $n - 1$ *orders, and if among the Wronskians of these functions taken* $n - 1$ *at a time there is at least one which does not vanish anywhere in* (a, b), *then, whenever* $W(y_1, \ldots, y_n) \equiv 0$, $y_1, \ldots, y_n$ *are linearly dependent throughout* (a, b),—and then to deduce the theorems with the help of another which he has found in other ways useful, namely, *If* $y_1, \ldots, y_n$ *have finite derivatives of the first* $n - 1$ *orders at every point of* (a, b), *then* $W(y_1, \ldots, y_n) \equiv 0$ *is a necessary and sufficient condition that* $y_1, \ldots, y_n$ *be linearly dependent throughout some subinterval.*

We venture to add that teachers still would find it hard to better the practice.

MORSE, H. M.; PFEIFFER, G. A.; GREEN, G. M. ($1916^5/_9$)

[Proof of a general theorem on the linear dependence of p analytic functions of a single variable. *Bull. American Math. Soc.*, xxiii. pp. 114–117.]

[Note on the linear dependence of analytic functions. *Bull. American Math. Soc.*, xxiii. pp. 117–118.]

[On the linear dependence of functions of one variable. *Bull. American Math. Soc.*, xxiii. pp. 118–122.]

The first of these writers on linear dependence brings forward a new Wronskian-like determinant formed from n analytic

functions $\phi_1, \phi_2, \ldots$ of a single variable. In it the r^{th} function as usual appropriates the r^{th} row. The columns on the other hand consist of μ sets, the first set containing $1+\lambda_1$ columns with the variable x_1, the second set containing $1+\lambda_2$ columns with the variable x_2, and so on, thus necessitating $n = \mu + \lambda_1 + \ldots + \lambda_\mu$, giving us the typical row

$$\phi_r(x_1),\ \phi_r'(x_1),\ \ldots,\ \phi_r^{(\lambda_1)}(x_1),\ \phi_r(x_2),\ \phi_r'(x_2),\ \ldots,\ \phi_r^{(\lambda_\mu)}(x_\mu),$$

and permitting description of the determinant as an aggregate of μ oblong Wronskian arrays. The associated test for linear dependence of the functions is that the determinant vanish identically in all of its variables.

The second writer shows that the test can be viewed as dependent on the Wronskian test; and the third writer extends the test so as to make it applicable when the functions are non-analytic.

COMPOSTO, S. (1916)

[Sui determinanti "Wronskiani fattoriali". *Giornale di Mat.*, liv. pp. 290–293.]

The so-called 'factorial Wronskian' differs from the Wronskian proper (1) in that the basic functions are restricted to being factorials like

$$x(x+\nu)(x+2\nu)\ldots(x+\overline{n-1}\cdot\nu) \quad \text{or} \quad (x,\nu)_n \ \text{ say:}$$

and (2) in that the derivational operation employed is not differentiation (though temporarily symbolized in the same way) but is such that when applied to $(x,\nu)_n$ gives $Cn(x,\nu)_{n-1}$ where C is a constant. The first result obtained is that

$$\begin{vmatrix} (x,\nu)_1 & (x,\nu)_1' & \ldots & (x,\nu)_1^{(n-1)} \\ (x,\nu)_2 & (x,\nu)_2' & \ldots & (x,\nu)_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ (x,\nu)_n & (x,\nu)_n' & \ldots & (x,\nu)_n^{(n-1)} \end{vmatrix}, \ \text{or } F(x,\nu) \text{ say,}$$

$$= 1!\ 2!\ \ldots\ (n-1)!\ (x,-\nu)_n$$

to which are attached several simple corollaries. More interest-

ing than these is the close analogue obtained to the differentiation-theorem in Wronskians.

COMPOSTO, S. (1919)

[Sui determinanti " Wronskiani fattoriali ". Nota 2ª. *Giornale di Mat.*, lvii. pp. 22–30.]

This second note may be said to concern the function $1/(x, \nu)_n$ or $(x, \nu)_{-n}$, as the first concerned $(x, \nu)_n$: and just as the main result in the first was the evaluation of the determinant $F(x, \nu)$, so here the chief problem is the evaluation of $F(x, \nu)_{-n}$. This is made to depend on another determinant, Δ_n, of the form which Hankel by means of a multiplier connected with a persymmetric. As a consequence much of the paper (pp. 24–29) falls to be considered elsewhere. When the main subject is resumed we learn that

$$F\left\{\frac{1}{(x, \nu)}\right\} = \frac{\pm \Delta_n}{(x, \nu)_1 (x, \nu)_2 \ldots (x, \nu)_n}$$

and ʳe two or three corollaries deduced therefrom as before.

LIST OF AUXILIARY WRITINGS

Mainly Applicational

$1890^{15}/_7$. Cels, J. Sur les équations différentielles linéaires ordinaires. *Comptes rendus . . . Acad. des Sci.* (Paris), cxi. pp. 98–100.

$1890^{8}/_{12}$. Cels, J. Sur une classe d'équations différentielles linéaires ordinaires. *Comptes rendus . . .*, cxi. pp. 879–881; cxii. pp. 985–988.

1891. Cels, J. Sur les équations différentielles linéaires ordinaires. *Annales de l'École Norm. . . .*, (3) viii. pp. 341–415.

$1895^{27}/_1$. Grünfeld, E. Ueber den Zusammenhang zwischen den Fundamentaldeterminanten einer linearen Differentialgleichung n^{ter} Ordnung und ihrer n Adjungirten. *Crelle's Journ.*, cxv. pp. 328–342.

1895/9. METZLER, G. F. Equations and variables associated with the linear differential equation. *Annals of Math.*, ix. pp. 171–178.

1908. WALLENBERG, G. Zur Theorie der homogenen linearen Differentialgleichungen. *Sitzungsb. d. Berliner Math. Ges.*, viii. pp. 22–26.

1914. GREEN, G. M. On completely integrable systems of homogeneous linear partial differential equations. *Bull. American Math. Soc.*, xxi. pp. 162–163.

1915 30/10. GREEN, G. M. The linear dependence of functions of several variables. *Bull. American Math. Soc.*, xxii. pp. 162, 165.

CHAPTER X

BORDERED DETERMINANTS, FROM 1900 TO 1920

The subject of bordered determinants has received only a very little more attention than in the preceding twenty-year period: and it has further to be noted that still fewer of the increased number of writings contribute to the advancement of the general theory. In no chapter, indeed, are we more concerned with the mere evaluation and discussion of special instances. As a help, therefore, towards the saving of space, we have refrained from separation in a considerable number of cases where an unbordered determinant and the corresponding bordered determinant are dealt with together in the original. No loss will come of this if the reader will now refer to Chapter I under the dates $1914^{20}/_1$, $1917^{21}/_3$; to Chapter IV under 1900, $1912/_6$, $1915^1/_1$, $1916^1/_2$, $1916^1/_{11}$, $1918^1/_5$; to Continuants under $1902^8/_9$; and to Chapter XI under $1911^{15}/_3$.

KNESER, A. ($1901^{23}/_7$)

[Aufgaben 12, 13. *Archiv d. Math. u. Phys.*, (3) i. pp. 368–369; iii. pp. 78–80.]

This calls one's mind back to the early days of the 'bordered discriminant' (*Hist.*, iii. pp. 432–433): and not quite fruitlessly as attention is thereby drawn to an early contribution of Richelot's which we had not formerly noticed,* and to a proof of it by Gundelfinger in his edition of Hesse's *Vorlesungen* published in 1876 (p. 454).

KOWALEWSKI, G. (1909)

[Einführung in die Determinantentheorie. iv + 550 pp. Leipzig.]

The space devoted to bordered determinants (pp. 89–99) is

* *Astron. Nachrichten*, xlviii. (1858) No. 1146.

rather encroached on by a second proof of Arnaldi's general expansion-theorem (*Hist.*, iv. pp. 432–433). This extends to four pages, and is based on multiplying the given determinant by the adjugate of its first r-line minor, and then this product by the adjugate of the last n-line minor, both multipliers being of course raised to the $(r+n)^{\text{th}}$ order.

MUIR, T. ($1914^{20}/_{2}$)

[The determinant of the sum of a square matrix and its conjugate. *Messenger of Math.*, xliii. pp. 184–192.]

The three concluding pages here concern the bordering of a duplicant. The case first considered is where the bordering line is one of the rows of the basic determinant, the result for the 4^{th} order being

$$\begin{vmatrix} \cdot & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_1+a_1 & a_2+b_1 & a_3+c_1 & a_4+d_1 \\ a_2 & b_1+a_2 & b_2+b_2 & b_3+c_2 & b_4+d_2 \\ a_3 & c_1+a_3 & c_2+b_3 & c_3+c_3 & c_4+d_3 \\ a_4 & d_1+a_4 & d_2+b_4 & d_3+c_4 & d_4+d_4 \end{vmatrix}$$

$$= -\frac{1}{a_1}\cdot\text{dupl. of}\begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ |a_1c_2| & |a_1c_3| & |a_1c_4| \\ |a_1d_2| & |a_1d_3| & |a_1d_4| \end{vmatrix},$$

where instead of $-1/a_1$ we should have $-1/a_1{}^{n-3}$ if the basic determinant had been of the n^{th} order. It is next shown that the result would have been the same if the bordering line had been

$$\cdot \quad a_1 \quad b_1 \quad c_1 \quad d_1;$$

and there are other results.

CORPUT, J. G. v. D. (1916)

[Vraagstuk 45. *Wiskundige Opgaven*, xii. pp. 111–113.]

A determinant got by a very special bordering of

$$|\sin\alpha_1 \ \sin 2\alpha_2 \ldots \sin n\alpha_n|$$

is here shown to be 0.

MUIR, T. ($1918^{10}/_{6}$)

[Note on the representation of the expansion of a bordered determinant. *Messenger of Math.*, xlviii. pp. 23–32.]

The representation brought into notice is in the form of a bilinear (or bipartite) function, and is illustrated by the case in which the order of the basic determinant is 5 and the breadth of the border 3, as follows:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{15} & \theta_{11} & \theta_{12} & \theta_{13} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{51} & a_{52} & \cdots & a_{55} & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{11} & \phi_{21} & \cdots & \phi_{51} & \cdot & \cdot & \cdot \\ \phi_{12} & \phi_{22} & \cdots & \phi_{52} & \cdot & \cdot & \cdot \\ \phi_{13} & \phi_{23} & \cdots & \phi_{53} & \cdot & \cdot & \cdot \end{vmatrix} =$$

$$(-1)^3 \begin{array}{c} (123)_\theta \quad (124)_\theta \quad \ldots \quad (345)_\theta \\ \hline \left.\begin{array}{cccc} \begin{vmatrix} 45 \\ 45 \end{vmatrix} & -\begin{vmatrix} 35 \\ 45 \end{vmatrix} & \cdots & \begin{vmatrix} 12 \\ 45 \end{vmatrix} \\ -\begin{vmatrix} 45 \\ 35 \end{vmatrix} & \begin{vmatrix} 35 \\ 35 \end{vmatrix} & \cdots & \begin{vmatrix} 12 \\ 35 \end{vmatrix} \\ \cdot & \cdot & \cdot & \cdot \\ \begin{vmatrix} 45 \\ 12 \end{vmatrix} & -\begin{vmatrix} 35 \\ 12 \end{vmatrix} & \cdots & \begin{vmatrix} 12 \\ 12 \end{vmatrix} \end{array} \right| \end{array} \begin{array}{l} (123)_\phi \\ (124)_\phi \\ \\ (345)_\phi \end{array}$$

where $(123)_\theta$ stands for $|\theta_{11}\,\theta_{22}\,\theta_{33}|$, $(245)_\phi$ for $|\phi_{21}\,\phi_{42}\,\phi_{53}|$,

and $\begin{vmatrix} 45 \\ 45 \end{vmatrix}$ for $|a_{44}\,a_{55}|$.

It is pointed out that the elements forming the square on the right are exactly the elements of the second compound of the basic determinant $|a_{15}|$, and that as a short notation for this compound is

$$\big|\,|a_{15}|\,\big|_2$$

the writing of the right-hand member may be simplified.

Pains is next taken to simplify the exposition of Arnaldi's theorem above referred to with a view to popularizing it. The expansion is first described as being *an aggregate of binary products, the first factor of each of which is a minor of a fixed coaxial minor*

of the given determinant, and the second factor a determinant got by bordering the complementary of the said minor. For example, the given determinant being $|a_1b_2c_3d_4e_5|$, the fixed coaxial minor $|a_1b_2c_3|$, and consequently the minor to be bordered $|d_4e_5|$, we have

$$\begin{aligned} |a_1b_2c_3d_4e_5| &= |a_1b_2c_3| \; |d_4e_5| \\ &+ \left\{ |a_1b_2| \begin{vmatrix} . & c_4 & c_5 \\ d_3 & d_4 & d_5 \\ e_3 & e_4 & e_5 \end{vmatrix} - \ldots + |b_2c_3| \begin{vmatrix} . & a_4 & a_5 \\ d_1 & d_4 & d_5 \\ e_1 & e_4 & e_5 \end{vmatrix} \right\} \\ &+ \left\{ a_1 \begin{vmatrix} . & . & b_4 & b_5 \\ . & . & c_4 & c_5 \\ d_2 & d_3 & d_4 & d_5 \\ e_2 & e_3 & e_4 & e_5 \end{vmatrix} - \ldots + c_2 \begin{vmatrix} . & . & a_4 & a_5 \\ . & . & b_4 & b_5 \\ d_1 & d_2 & d_4 & d_5 \\ e_1 & e_2 & e_4 & e_5 \end{vmatrix} \right\}, \end{aligned}$$

where (1) we begin with the highest minor of $|a_1b_2c_3|$, namely, $|a_1b_2c_3|$ itself, then give an assemblage of terms in which the first factors are the 2-line minors of $|a_1b_2c_3|$ and lastly an assemblage in which the first factors are the 1-line minors, i.e. the elements: (2) we begin with a border of no breadth for $|d_4e_5|$, then affix a 1-line border, and lastly a 2-line border. An alternative mode of getting the second factors in practice is also given, and this leads to a full formulation of the theorem in words, namely: *If in any determinant* P *and* Q *be complementary coaxial minors, and if every possible minor of* P *be taken and multiplied by its complementary in the given determinant, the said complementary being so altered that the complementary of* Q *in it has only zero elements, the aggregate of the products so obtained is equal to the given determinant.*

LIST OF AUXILIARY WRITINGS

Mainly Applicational

1901. Gravé, D. A. On some applications of determinants (In Russian). *Mat. Sbornik* . . . (Moscow), xxii. pp. 243–253.

1911¹/₁₁. Bateman, H. On a set of kernels whose determinants form a Sturmian sequence. *Bull. American Math. Soc.*, (2) xviii. pp. 179–182.

CHAPTER XI

THE LESS COMMON SPECIAL FORMS, FROM 1884 TO 1919

The number of writings classifiable under this heading continues to increase at a surprising rate. The number here falling to be dealt with, 112, is almost quite a half more than the corresponding number for the immediately preceding twenty-year period: and this notwithstanding the fact that it has been made about a score less through the withdrawal, for separate treatment, of those concerned with Fredholm's determinant. About three-fourths of those thus remaining deal with forms that have already presented themselves somewhere in one or more of the preceding volumes. No other form, however, comes near Fredholm's in the strength of its claim for a similar preference. The three nearest are but poor seconds—determinants with Definite Integrals for elements, determinants whose elements are Combinatory Numbers, and determinants that are wholly or partially Zero-axial. Still seldomer recalled to notice are Hermitants, Block determinants, Unisignants, Permanents, Voigt's skew determinant, Duplicants, Group determinants, and so forth. Of course towards the end of the list the interest suddenly increases, as the forms to be considered are quite new or the subject-matters come up in new connections. Equally of course they vary greatly in importance, from a 3-line determinant set for resolution into factors up to a fresh n-line type that calls for and is sure to receive further study.

SEELHOFF, P. (1884)

[Ueber allgemeine und absolute Permutationen. *Archiv d. Math. u. Phys.*, (2) i. pp. 97–101.]

This is non-determinantal, but throws a sidelight on the problem of finding ψ the number of terms in a zero-axial determinant, this being the same as the number of the author's 'absolute permutations' (*Hist.*, iii. pp. 463–464).

PEIN, A. ($1889/_4$)

[Aufstellung von n Königinnen auf dem Schachbrett von n^2 Feldern derart, dass keine von einer andern geschlagen werden kann. Sch. Progr. 62 pp. Leipzig.]

This lengthy account of the problem of the n queens is valuable in the first place from a purely historical point of view. It is also of importance in that, with the help of determinants as used by Günther (*Hist.*, iii. pp. 486–487), the author succeeds in finding the number of solutions for the case where n is 9 to be 352, and for the case where n is 10 to be 724, the number agreeing in the former case with what T. B. Sprague otherwise arrived at in the following year.*

CAPELLI, A. ($1893/_8$): CAZZANIGA, T. ($1900^{27}/_3$)

[Alcune formole relative alle operazione di polare. *Giornale di Mat.*, xxxii. pp. 376–380: also in French in the *Proceed. Math. Congress* of Chicago.]

[Aggiunte ad una mia nota intorno ai determinanti. *Rendic. . . . Ist. Lombardo* . . . (Milano), (2) xxxiv. pp. 176–179.]

These are both continuations of work already reported on (*Hist.*, iv. pp. 462–463, 476–477, 488–489). The type of determinant mainly involved and discussed in the original papers is that in which the axial elements are in equidifferent progression. The present papers naturally incline more to the applicational side: and to a greater extent the same is true of two other related papers by J. Deruyts and S. Minnetola. The latter have consequently been transferred to the Auxiliary List at the end of the chapter.

KÜRSCHÁK, J. (1895)

[Hunyady Jenö egyik determinans-tételéröl. *Math. és Phys. Lapok*, iv. pp. 1–6.]

The theorem of Hunyady's which Kürschák refers to in his title is

* *Proceed. Edinburgh Math. Soc.*, viii. p. 40.

$$\begin{vmatrix} |y_bz_c|^2 & |z_bx_c|^2 & |x_by_c|^2 & |z_bx_c|\,|x_by_c| & |x_by_c|\,|y_bz_c| & |y_bz_c|\,|z_bx_c| \\ |y_cz_a|^2 & |z_cx_a|^2 & |x_cy_a|^2 & |z_cx_a|\,|x_cy_a| & |x_cy_a|\,|y_cz_a| & |y_cz_a|\,|z_cx_a| \\ |y_az_b|^2 & |z_ax_b|^2 & |x_ay_b|^2 & |z_ax_b|\,|x_ay_b| & |x_ay_b|\,|y_az_b| & |y_az_b|\,|z_ax_b| \\ |y_az_d|^2 & |z_ax_d|^2 & |x_ay_d|^2 & |z_ax_d|\,|x_ay_d| & |x_ay_d|\,|y_az_d| & |y_az_d|\,|z_ax_d| \\ |y_bz_d|^2 & |z_bx_d|^2 & |x_by_d|^2 & |z_bx_d|\,|x_by_d| & |x_by_d|\,|y_bz_d| & |y_bz_d|\,|z_bx_d| \\ |y_cz_d|^2 & |z_cx_d|^2 & |x_cy_d|^2 & |z_cx_d|\,|x_cy_d| & |x_cy_d|\,|y_cz_d| & |y_cz_d|\,|z_cx_d| \end{vmatrix},$$

or H say,

$$= - \,|x_by_cz_d|^2\, |x_cy_dz_a|^2\, |x_dy_az_b|^2\, |x_ay_bz_c|^2.$$

It was arrived at originally on p. 12 of Hunyady's first paper of 1880 (*Hist.*, iv. pp. 202–204), a paper which, however, it is best to view not by itself but as one of a considerable number of papers which originated with Hunyady's of 1875 and to which Scholtz, Mertens, and Pasch also contributed (*Hist.*, iii. pp. 191–192, 195–196, 202–207; iv. pp. 204–209). Of this theorem Kürschák at the outset furnishes a fresh proof. Taking another special determinant

$$\begin{vmatrix} x_a & \cdot & \cdot & \cdot & z_a & y_a \\ \cdot & y_b & \cdot & z_b & \cdot & x_b \\ \cdot & \cdot & z_c & y_c & x_c & \cdot \\ x_d & \cdot & \cdot & \cdot & z_d & y_d \\ \cdot & y_d & \cdot & z_d & \cdot & x_d \\ \cdot & \cdot & z_d & y_d & x_d & \cdot \end{vmatrix}, \text{ or } \mu \text{ say,}$$

which is readily shown to be transformable into

$$\begin{vmatrix} \cdot & |z_dx_a| & |y_dx_a| \\ |z_by_d| & \cdot & |x_by_d| \\ |y_dz_c| & |x_dz_c| & \cdot \end{vmatrix},$$

he multiplies H by μ in its 6-line form and succeeds in showing that the product is resolvable into

$$- \,|x_ay_bz_c|^2\, |x_by_cz_d|^2\, |x_cy_dz_a|^2\, |x_dy_az_b|^2 \cdot \mu,$$

and thus that division by μ is all that is further required.

Following on this comes a result of Kürschák's own, namely,

$$\begin{vmatrix} x_a x_{a'} & y_a y_{a'} & z_a z_{a'} & y_a z_{a'} + z_a y_{a'} & z_a x_{a'} + x_a z_{a'} & x_a y_{a'} + y_a x_{a'} \\ x_b x_{b'} & y_b y_{b'} & z_b z_{b'} & y_b z_{b'} + z_b y_{b'} & z_b x_{b'} + x_b z_{b'} & x_b y_{b'} + y_b x_{b'} \\ x_c x_{c'} & y_c y_{c'} & z_c z_{c'} & y_c z_{c'} + z_c y_{c'} & z_c x_{c'} + x_c z_{c'} & x_c y_{c'} + y_c x_{c'} \\ x_d x_{d'} & y_d y_{d'} & z_d z_{d'} & y_d z_{d'} + z_d y_{d'} & z_d x_{d'} + x_d z_{d'} & x_d y_{d'} + y_d x_{d'} \\ x_a x_c & y_a y_c & z_a z_c & y_a z_c + z_a y_c & z_a x_c + x_a z_c & x_a y_c + y_a x_c \\ x_b x_d & y_b y_d & z_b z_d & y_b z_d + z_b y_d & z_b x_d + x_b z_d & x_b y_d + y_b x_d \end{vmatrix},$$

or K say,

$$= | x_c y_d z_{a'} | \, | x_d y_a z_{b'} | \, | x_a y_b z_{c'} | \, | x_b y_c z_{d'} | \\ - | x_{a'} y_b z_c | \, | x_{b'} y_c z_d | \, | x_{c'} y_d z_a | \, | x_{d'} y_a z_b |.$$

It is obtained in a quite similar manner, K being multiplied by H in determinant form and then H in its factorial form removed from the product. Special cases of the theorem are then duly noted as being already known, and the requisite references given to papers by Scholtz and Hunyady.

We may note further for ourselves that the independent variables of H form a 3-by-4 array

$$\begin{matrix} x_a & x_b & x_c & x_d \\ y_a & y_b & y_c & y_d \\ z_a & z_b & z_c & z_d, \end{matrix}$$

the 2-line minors of which appear on the left hand of H and the 3-line minors on the right. In K there is involved an additional 3-by-4 array of variables, this second array, however, not playing so full a part as the other—a fact that will be clear on observing that the row-sums of K are expressible as products, and that the products in the case of the 5^{th} and 6^{th} rows are independent of the second array, being

$$(x_a + y_a + z_a)(x_c + y_c + z_c) \quad \text{and} \quad (x_b + y_b + z_b)(x_d + y_d + z_d)$$

respectively.

FROBENIUS, G. (1896)

[Ueber die Primfactoren der Gruppendeterminante. *Sitzungsb.... Ges. d. Wiss.* (Berlin), 1896, pp. 1343–1382.]

This communication is a sequel to that of the same year on ‘group-characters’ to which we had occasion to refer (*Hist.*, iv.

pp. 390–391) because of the statement made in it regarding Dedekind's early recognition of the importance of what he (Frobenius) here calls the 'Gruppendeterminante', and more particularly because of the concrete instance of such a determinant which Dedekind had resolved into factors. Now the author no longer deals with mere instances of this peculiar associate of a group, but with the general entity and its properties. Having defined it by means of the 'characters' of the group he studies it mainly in regard to its resolvability and to the bearing, on group-theory, of the number and nature of the factors that may be obtainable.

SPRAGUE, T. B. ($1899^{10}/_2$)

[On the eight-queens problem. *Proceed. Edinburgh Math. Soc.*, xvii. pp. 43–68.]

This paper, which is of considerable general value, has a special interest for us because of its clearly-drawn comparison (pp. 49–61) between the methods of Gauss and Günther, the latter of whom, as we have seen (*Hist.*, iii. pp. 486–487), differed from his predecessors in using determinants. At the close he gives not only the number of solutions (2680) for the case where n is 11, but a table of the actual solutions themselves.

FRATTINI, G. ($1899^3/_{10}$)

[Quistione 158. *Periodico di Mat.*, xv. p. 83.]

A simple property of unit determinants of the second order.

SCHMIDT, H. ($1900/_1$)

[Beweis eines Determinantensatzes. *Math.-naturw. Mitteilungen* (Würtemberg), (2) ii. pp. 20–21.]

The subject here is the determinant

$$\begin{vmatrix} p^r - q^r & rp^{r-1} & rq^{r-1} \\ p^s - q^s & sp^{s-1} & sq^{s-1} \\ p^t - q^t & tp^{t-1} & tq^{t-1} \end{vmatrix}$$

which, on the face of it, is seen to be divisible by $(p-q)^2$. The author's interesting result is that, so long as r, s, t are different positive integers, it is exactly divisible by $(p-q)^4$. A convenient expression for the cofactor would be welcome.

JOLY, C. J. (1900)

[On quaternion determinants. *Hamilton's Elements of Quaternions.* Second Edition; ii. pp. 361–363, 382, 393.]

The convention here acted on in regard to the order in which the elements are to be taken to form the terms of a quaternion determinant is naturally the same as in the editor's paper of 1896 (*Hist.*, iv. p. 490) and different from that in Cayley's original paper of 1845 (*Hist.*, ii. pp. 459–461); so that

$$\begin{vmatrix} p & q \\ p' & q' \end{vmatrix} \neq pq' - p'q \text{ but} = pq' - qp',$$

and
$$\neq \begin{vmatrix} p & q \\ xp+p' & xq+q' \end{vmatrix} \text{ but} = \begin{vmatrix} p & xp+q \\ p' & xp'+q' \end{vmatrix}.$$

Supplementary to Cayley's contribution note is taken that when the multiplier is a scalar determinant we have

$$\begin{vmatrix} p\,q \\ p'q' \end{vmatrix} \cdot \begin{vmatrix} x\,y \\ x'y' \end{vmatrix} = \begin{vmatrix} p\,x+q\,y & p\,x'+q\,y' \\ p'x+q'y & p'x'+q'y' \end{vmatrix} = \begin{vmatrix} p\,x+q\,x' & p\,y+q\,y' \\ p'x+q'x' & p'y+q'y' \end{vmatrix};$$

and the effect of the vanishing of a sum of scalar multiples of the elements of a row is discussed in connection with the determinants

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}, \quad \begin{vmatrix} p_1 & q_1 & r_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}, \quad \begin{vmatrix} p_1 & q_1 & r_1 \\ p_1 & q_1 & r_1 \\ p_1 & q_1 & r_1 \end{vmatrix}.$$

SIBIRANI, F. ($1900/_2$)

[Su alcuni determinanti. *Periodico di Mat.*, (2) ii. pp. 247–252.]

The plan of this paper is somewhat unusual, as if the author had set himself the task of framing a collection of determinants specifiable by means of a single typical element and then evalu-

ating them. It is at any rate the case that he simply takes the following five unconnected functions of r and s,

$$f(r)\cdot\phi(s),\quad f(r)+\phi(s),\quad \{f(r)\}^s,\quad (r+s)^m,\quad \frac{r^m-s^m}{r-s},$$

and proceeds to obtain the values of the corresponding determinants $|f(r)\cdot\phi(s)|_n$, . . .

Most of the results are already known or are easily calculable. It is of course the last two functions of r and s that give him most trouble, and the expressions there reached are not always very concise.

FERBER, F. (1900/$_7$)

[Application du symbole des déterminants positifs. *Bull. Soc. Math. de France*, xxviii. pp. 128–130.]

This is virtually a continuation of the author's paper of the preceding year (*Hist.*, iv. pp. 459–460), and gives two additional examples of the use of permanents. The example which concerns the sum of the p^{th} powers of the first n integers is more curious than helpful.

SIBIRANI, F. (1900^1/$_{12}$), (1902/$_1$)

[Un notevole specchio di numeri. *Periodico di Mat.*, xvi. pp. 278–284.]

[Sopra una classe di determinanti. *Periodico di Mat.*, xvii. pp. 316–319.]

The array referred to in the first of these titles is

$$\begin{array}{cccccc}
1 \\
1 & 1 \\
2 & 3 & 1 \\
6 & 11 & 6 & 1 \\
24 & 50 & 35 & 10 & 1 \\
120 & 274 & 225 & 85 & 15 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

the law of formation of its elements being

$$a_{rs} = (r-1)a_{r-1,\,s} + a_{r-1,\,s-1}.$$

A determinant whose elements are related to the elements of this array is evaluated, but the author wisely gives also the following independent statement of the result: *If* $S_{h,\,n}$ *denotes the sum of the products of the numbers* 1, 2, 3, . . . , n — 1 *taken* h *at a time, then*

$$\begin{vmatrix} S_{n,\,n+1} & S_{n-1,\,n+1} & \ldots & S_{n-k+1,\,n+1} \\ S_{n+1,\,n+2} & S_{n,\,n+2} & \ldots & S_{n-k+2,\,n+2} \\ \cdot & \cdot & \cdot & \cdot \\ S_{n+k-1\,\,n+k} & S_{n+k-2,\,n+k} & \ldots & S_{n,\,n+k} \end{vmatrix} = (n!)^k.$$

The second paper re-establishes this and arrives at a companion result, which, however, is essentially the same as Janni's of 1876 (*Hist.*, iii. p. 460).

LAISANT, C. A. ($1901^{1}/_{5}$)

[Question 445. *Nouv. Annales de Math.*, (4) i. p. 232.]

This problem which had been standing unsolved since 1858 is now interpreted to be an equivalent of 'the problem of the n queens', namely, to find the number of terms of a determinant that remain after every term has been excluded which contains two elements whose joining line is coincident with or parallel to a diagonal. It is also pointed out that another variant of the problem had received attention in *Nouv. Annales*, (2) iii. p. 187 and (2) xx. pp. 473–480.

AUTONNE, L. ($1901^{22}/_{7}$)

[Sur l'hermitien. *Comptes rendus . . . Acad. des Sci.* (Paris), cxxxiii. pp. 209–210: *Rendic. del Circ. Mat.* (Palermo), xvi. pp. 104–128.]

The subject here is Hermite's bilinear forms but not these forms in general: the writer confines himself to those that are

'positive definite', and it is those only that he calls "Hermitians'. The determinant of the form—the Hermitant—is of course always in evidence throughout the exposition, and this is one reason for the paper being drawn attention to: but there is another, for almost equal space is given in it to what is similarly named the 'Hermitienne' (that is, the Hermitian substitution) the determinant of which is the same Hermitant. The calculus employed is that of Cayleyan matrices as expounded and developed in Frobenius' classical memoir of 1877 (*Crelle's Journ.*, lxxxiv. pp. 1–63).*

GAVRILOVITCH, B. ($1901^{15}/_{9}$)

[On some properties of a special determinant (In Serbian). *Proceed. Servian Acad. Sci.*, lxiii. pp. 241–254.]

The determinant here considered somewhat lengthily is that in which the diagonal elements and all the elements on one side of the diagonal are equal to x.

VAHLEN, K. T. (1901): NEUBERG, J. ($1901/_{10}$)

[Sul teorema di Brioschi degli 8 quadrati. *Giornale di Mat.*, xxxix. pp. 181–184.]

[Question 14985. *Educ. Times*, liv. p. 424; lv. p. 38; or *Math. from Educ. Times*, (2) ii. p. 32; *Mathesis*, xxxix. p. 195.]

Vahlen's paper has little determinantal interest: and its reference to Brioschi should be received with caution (*Hist.*, iii. p. 281: also present chapter under Muir, $1907^{1}/_{9}$).

Neuberg's result, established by Muir, is

$$\left| x_1+x_2-x_3-x_4 \quad y_1+y_3-y_4-y_2 \quad x_1x_4-x_2x_3+y_1y_4-y_2y_3 \right| = 2\begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{vmatrix}.$$

* For continuations of this research of Autonne's see the Auxiliary List under 1903 and 1913.

CREPAS, A. $(1902/_2)$

[Determinanti figurati e determinanti speciali. *Periodico di Mat.*, xvii. pp. 161–175.]

From any determinant D a 'figurate' determinant, as here defined, is formable by increasing the elements of any row by 1, or $(q)_0$ say, of the next row by $(q+1)_1$, of the next row by $(q+2)_2$, and so on to the last row. This implies that if D be of the n^{th} order there are n such associated determinants, whose values will ordinarily depend not only on D and q but also on the number of D's row with which the addition of the figurate integers begins. The writer's lengthy investigation of properties (pp. 161–167) starts with Sardi's device of 'raising the order to simplify the elements' (*Hist.*, iii. p. 125), the immediate result being a determinant derivable from D in another way, namely, by bordering. In like detail are next dealt with penesymmetric determinants and others of a very special type (*Hist.*, iii. pp. 100–101, 125–126): and lastly (pp. 174–175) the effect of augmenting every element of an orthogonant by 1 is considered, and the effect of augmenting each element of a null determinant by the reciprocal of the cofactor of the said element.

NANSON, E. J. $(1902/_7)$

[Question 15146. *Educ. Times*, lv. pp. 305, 515–516: or *Math. from Educ. Times*, (2) iii. pp. 108–109.]

The result here is

$$|\, B_1C_1 \cdot C_2A_2 \cdot A_3B_3 \,| = -\,|\, a_1b_2c_3 \,|^2 \cdot |\, b_1c_1 \cdot c_2a_2 \cdot a_3b_3 \,|$$

on the understanding that $|\, A_1B_2C_3 \,|$ is the adjugate of $|\, a_1b_2c_3 \,|$ (cf. *Hist.*, iv. p. 67). A companion result is given in Muir's second paper of 1907 reported on below, namely,

$$\begin{aligned} &|\, A_1{}^2B_2{}^2C_3{}^2 \,| - |\, A_1A_2 \cdot B_2B_3 \cdot C_3C_1 \,| \\ &\qquad = |\, a_1b_2c_3 \,|^2 \cdot \{|\, a_1{}^2b_2{}^2c_3{}^2 \,| - |\, a_1a_2 \cdot b_2b_3 \cdot c_3c_1 \,|\}. \end{aligned}$$

TUCKER, R. (1902[1]/9)

[Question 15195. *Educ. Times*, lv. p. 396: or *Math. from Educ. Times*, (2) v. pp. 119–120.]

The result, when altered a little in the statement, is

$$\begin{vmatrix} \beta\gamma & \gamma(\gamma-\beta) & \beta(\beta-\gamma) \\ \gamma(\gamma-\alpha) & \gamma\alpha & \alpha(\alpha-\gamma) \\ \beta(\beta-\alpha) & \alpha(\alpha-\beta) & \alpha\beta \end{vmatrix} = \alpha\beta\gamma \begin{vmatrix} \alpha & \gamma-\beta & \beta-\gamma \\ \gamma-\alpha & \beta & \alpha-\gamma \\ \beta-\alpha & \alpha-\beta & \gamma \end{vmatrix}$$

$$= \alpha\beta\gamma(\alpha+\beta-\gamma)(\beta+\gamma-\alpha)(\gamma+\alpha-\beta).$$

MUIR, T. (1902[8]/9)

[The generating function of the reciprocal of a determinant. *Transac. R. Soc. Edinburgh*, xl. pp. 615–629.]

In working out the generalizations of Jacobi's theorem of 1829 a peculiar form of determinant comes into prominence which has the rare property of being expressible as an aggregate of bilinears: for example,

$$\begin{vmatrix} U_2 - b_2x_2 & -b_3x_3 & -b_4x_4 \\ -c_2x_2 & U_3 - c_3x_3 & -c_4x_4 \\ -d_2x_2 & -d_3x_3 & U_4 - d_4x_4 \end{vmatrix}$$

where

$$U_2 = b_1x_1 + \dots + b_4x_4, \quad U_3 = c_1x_1 + \dots + c_4x_4,$$
$$U_4 = d_1x_1 + \dots + d_4x_4$$

is equal to

$$x_1 \cdot \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline b_1c_1d_1 & b_1c_2d_1 & b_1c_1d_3 & b_1c_4d_1 & x_1 \\ b_1c_1d_2 & b_1c_2d_2 & b_1c_2d_3 & b_1c_4d_2 & x_2 \\ b_3c_1d_1 & b_3c_1d_2 & b_3c_1d_3 & b_4c_1d_3 & x_3 \\ b_4c_1d_1 & b_4c_2d_1 & b_3c_4d_1 & b_4c_4d_1 & x_4, \end{array}$$

the first step in the proof being the addition of the elements of each row. The character of the general theorem reached will be understood from the statement of the case of the 4^{th} order: *If* $\mathrm{U}_2, \ldots, \mathrm{U}_5$ *stand for* $\mathrm{b}_1\mathrm{x}_1 + \ldots + \mathrm{b}_5\mathrm{x}_5, \ldots, \mathrm{e}_1\mathrm{x}_1 + \ldots + \mathrm{e}_5\mathrm{x}_5$, *there are twenty-four determinants of the type*

$$\begin{vmatrix} U_2 - b_2x_2 & -b_3x_3 & -b_4x_4 & -b_5x_5 \\ -c_2x_2 & U_3 - c_3x_3 & -c_4x_4 & -c_5x_5 \\ -d_2x_2 & -d_3x_3 & U_4 - d_4x_4 & -d_5x_5 \\ -e_2x_2 & -e_3x_3 & -e_4x_4 & U_5 - e_5x_5 \end{vmatrix}$$

which are all presentable in the form $\mathrm{x}_1\mathrm{F}_1$, *where* F_1 *is a complete cubic in* $\mathrm{x}_1, \mathrm{x}_2, \ldots, x_5$ *consisting of* 5^3 *positive terms, namely, in the case of the determinant specified, five terms of the form* $\mathrm{bcde} \cdot \mathrm{x}_1{}^3$, *twenty compound terms of the form*

$$x_1{}^2x_2 \cdot (b_1c_1d_1e_2 + b_1c_1d_2e_1 + b_1c_2d_1e_1),$$

and ten compound terms of the form

$$x_1x_2x_3 \cdot (b_1c_1d_2e_3 + b_1c_1d_3e_2 + b_1c_2d_1e_3 + b_1c_2d_3e_1 + b_3c_1d_1e_2 + b_3c_1d_2e_1).$$

A closely associated fact is also insisted on, namely, that the square array of each bilinear can be made inverso-symmetric: for example, the 3-line determinant above can be written

$$\begin{array}{cccc|l} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & \frac{c_2}{c_1} & \frac{d_3}{d_1} & \frac{e_4}{c_1} & x_1 \cdot b_1c_1d_1 \\ \frac{c_1}{c_2} & 1 & \frac{d_3}{d_2} & \frac{c_4}{c_2} & x_2 \cdot b_1c_2d_2 \\ \frac{d_1}{d_3} & \frac{d_2}{d_3} & 1 & \frac{b_4}{b_3} & x_3 \cdot b_3c_1d_3 \\ \frac{c_1}{c_4} & \frac{c_2}{c_4} & \frac{b_3}{b_4} & 1 & x_4 \cdot b_4c_4d_1. \end{array}$$

No reference is made to the resemblance between the new determinants and Sylvester's unisignant of 1859 (*Hist.*, ii. pp. 456–459): but in connection with the question of the number of terms note is formally taken that *in the final expansions of the determinants*

$$\begin{vmatrix} a_1 + \ldots + a_z & -a_1 & -a_2 & \ldots \\ -\beta_1 & \beta_1 + \ldots + \beta_z & -\beta_2 & \ldots \\ -\gamma_1 & -\gamma_2 & \gamma_1 + \ldots + \gamma_z & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n,$$

$$\begin{vmatrix} a_w & -a_1 & -a_2 & \ldots \\ \beta_w & \beta_1 + \ldots + \beta_z & -\beta_2 & \ldots \\ \gamma_w & -\gamma_2 & \gamma_1 + \ldots + \gamma_z & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n$$

all the terms are positive, the number in the former being

$$(z - n + 1)(z + 1)^{n-1}$$

and in the latter $(z + 1)^{n-1}$.

HUBER, H. T. ($1902^{20}/_{11}$)

[Z teoryi wyznaczników. *Wiadomosci mat.*, vi. pp. 317–324.]

The subject here dealt with is the number of terms in a determinant having a number of zeros in the diagonal. The familiar formula for the case where *all* the diagonal elements are zeros is first carefully gone into (pp. 317–321), the difference between $\psi(n)$ and $n!e^{-1}$ being drawn attention to and emphasized by comparing the first five values of the former with those of the latter, namely,

	1,	2,	9,	44,	265,
with	·7,	2·2 . . . ,	8·8 . . . ,	44·1 . . . ,	264·8 . . . .

The remainder of the paper follows on Weyrauch's of 1871 (*Hist.*, iii. p. 464).

SAALSCHÜTZ, L. ($1903^{14}/_5$)

[Einfache Determinantensätze. *Berichte d. phys.-ökon. Ges.* (Königsberg), 1903, pp. 8–9.]

The evaluations here made we are already in part familiar with, previous contributors being Zeipel, Janni, &c. (*Hist.*, iii. pp. 449, 460, &c.). We now take note of the generalization

$$\begin{vmatrix} a_0 & (a+1)_0 & (a+2)_0 & \ldots \\ b_1 & (b+1)_1 & (b+2)_1 & \ldots \\ c_2 & (c+1)_2 & (c+2)_2 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 1$$

where of course $a, b, c, \ldots$ are positive integers, and the fresh result

$$\begin{vmatrix} P(2k+1) & (m)_1 \cdot P(2k-1) & (m)_2 \cdot P(2k-3) & \ldots \\ P(2k+3) & (m+1)_1 \cdot P(2k+1) & (m+1)_2 \cdot P(2k-1) & \ldots \\ P(2k+5) & (m+2)_1 \cdot P(2k+3) & (m+2)_2 \cdot P(2k+1) & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n$$

$$= (2k-2m+1)^{n-1}(2k-2m-1)^{n-2} \ldots (2k-2m-2n+5)$$
$$\cdot P(2k+1) \cdot P(2k-1) \cdot P(2k-3) \ldots P(2k-2n+3),$$

where $P(2k+1)$ stands for $(2k+1)(2k-1)\ldots 3{\cdot}1$.

LAZARUS, A. ($1903/_6$)

[Questions 2596, 2623. *L'Intermédiaire des Math.*, x. pp. 150, 179–181.]

The question raised here is that of the possible existence of zero elements in corresponding places of a determinant and its square, or of a determinant and its adjugate. Apparently nothing came of the inquiry.

PERNA, A. ($1903/_8$)

[Intorno ad alcuni aggregati di coefficienti binomiali. *Giornale di Mat.*, xli. pp. 321–335.]

As a matter of fact the aggregates here dealt with are aggregates of *products of pairs of* binomial-coefficients; for example, the aggregate denoted by $A_{m,n,l}$ is

$$(m-2n)_l - (n)_1 \cdot (m-2n)_{l-2} + (n)_2 \cdot (m-2n)_{l-4} - \ldots$$

and the determinants evaluated have such aggregates for elements, for example:

$$\begin{vmatrix} A_{m,0,0} & A_{m,1,0} & \cdots & A_{m,\mu,0} \\ A_{m,0,2} & A_{m,1,2} & \cdots & A_{m,\mu,2} \\ \cdot & \cdot & \cdots & \cdot \\ A_{m,0,2\mu} & A_{m,1,2\mu} & \cdots & A_{m,\mu,2\mu} \end{vmatrix} = (-1)^{(m+1)_2} \cdot 2^M,$$

where M is the highest integer in $m^2/4$.

NANSON, E. J. (1903, 1904): MUIR, T. (1903)

[Questions 15375, 15426. *Educ. Times*, lvi. pp. 305, 439, 441; lvii. p. 345: or *Math. from Educ. Times*, (2) v. pp. 74–75.]

Nanson's initial equality here is

$$|a_1X_2Y_3Z_4|\,|f_1X_2Y_3Z_4| + |b_1X_2Y_3Z_4|\,|g_1X_2Y_3Z_4| + |c_1X_2Y_3Z_4|\,|h_1X_2Y_3Z_4| = 0$$

where $X_p, Y_p, Z_p = h_py - g_pz + a_pw,\ f_pz - h_px + b_pw,\ g_px - f_py + c_pw.$

This, with an eye to generalization, Muir proves by expressing each of the determinants as a product of two 4-by-6 arrays, his extended result being formulated as Question 15426. The following year the proposer himself indicates a mode of proof and in turn gives a further generalization.

GRACE, J. H. AND YOUNG, A. (1903)

[The Algebra of Invariants. vii + 384 pp. Cambridge.]

Besides being helpful generally in regard to the handling of determinants by the school of Clebsch and Gordan, this textbook may also be profitably consulted in regard to certain special forms known as 'operators', which we have repeatedly had to refer to under the names of Capelli and Cayley (*Hist.*, iv. pp. 57, 462–463, 470, . . .). As a succinct instance we may quote the theorem (§ 211):

If the Cayleyan operator

$$\begin{vmatrix} \dfrac{\partial}{\partial\xi_1} & \dfrac{\partial}{\partial\eta_2} & \dfrac{\partial}{\partial\zeta_3} \end{vmatrix} \textit{ be denoted by } \Omega,$$

and m *and* r *be positive integers, then*

$$\Omega^r\,|\xi_1\eta_2\zeta_3|^m \textit{ is a multiple of } |\xi_1\eta_2\zeta_3|^{m-r}.$$

MANDART, H. ($1903^1/_9$)

[Question 1402. *Mathesis*, (3) iii. pp. 140, 213–214.]

The determinant here said to have been evaluated 'par des calculs assez pénibles'

$$\begin{vmatrix} (a+1)(x+y) & 2(axy-1) & (a-1)(y-x) \\ 1-xy+b(x+y) & 2bxy+x+y & b(y-x) \\ c(x+y) & 2cxy+x-y & 1+xy+c(y-x) \end{vmatrix},$$

is well worth attention. Perhaps the best initial step is to change it into

$$\begin{vmatrix} 1 & x+y & 2xy & y-x \\ \cdot & x+y+a(x+y) & -2+a(2xy) & x-y+a(y-x) \\ \cdot & 1-xy+b(x+y) & x+y+b(2xy) & b(y-x) \\ \cdot & c(x+y) & x-y+c(2xy) & 1+xy+c(y-x) \end{vmatrix}$$

which is readily seen to be equal to

$$\begin{vmatrix} 1 & x+y & 2xy & y-x \\ -a & x+y & -2 & x-y \\ -b & 1-xy & x+y & \cdot \\ -c & \cdot & x-y & 1+xy \end{vmatrix}$$

whence the final result

$$2(x^2+1)(y^2+1)\{axy+b(x+y)-c(x-y)+1\}$$

may be got in a variety of ways.

OCCHIPINTI, R. ($1903/_{11}$)

[Su alcuni determinanti. *Periodico di Mat.*, (3) i. pp. 142–143.]

The purpose here is to examine the properties of a determinant whose rows are such that

$$(\text{row}_r)^2 = 0, \quad \text{row}_r \cdot \text{row}_s = 1.$$

In definition the new determinant is a sort of antithesis to the orthogonant: and its square is the circulant $C(0, 1, 1, 1, \ldots)_n$ whose value is $(-1)^{n-1}(n-1)$, being a case of Sylvester's cir-

culant of 1855 (*Hist.*, ii. pp. 406–407). The analogous properties obtained are not of marked interest: for example, corresponding to the equality $a_{rs} = A_{rs}$ in orthogonants there is found

$$a_{rs} \cdot |a_{1n}| = A_{r1} + A_{r2} + \ldots + A_{r,r-1} + A_{r,r+1} + \ldots + A_{rn}.$$

DICKSON, L. E. (1903/$_{12}$)

[A generalization of symmetric and skew symmetric determinants. *American Math. Monthly*, x. pp. 253–256.]

The general determinant here referred to is Hermite's of 1854 (*Hist.*, ii. p. 449), the last reference to which in the History is that concerning Studnička's paper of 1899 (*Hist.*, iv. p. 497). The special matter brought forward, however, is that more fully discussed by Muir in 1897 (*Hist.*, iv. pp. 492–493). The expressions obtained for the 3-line and 4-line determinants agree with Muir's: beyond this the means of comparison is not available.

STETSON, O. S. (1904, 1905)

[Note on the expansion of devertebrate determinants. *American Math. Monthly*, xi. pp. 166–168.]

[A short proof for the number of terms in a determinant which are independent of the elements of the principal diagonal. *American Math. Monthly*, xii. p. 84.]

The main contribution in the first paper here is a general proof of Muir's theorem of 1898 regarding the expansion of a partially invertebrate determinant.* It includes of course the case where the determinant to be expanded is wholly invertebrate, a theorem due really to Cunningham (*Hist.*, iv. p. 67). Attention also is given to Cayley's related theorem of 1847 (*Hist.*, ii. p. 42).

DIXON, A. L. (1904^2/$_5$)

[Generalizations of Legendre's formula $KE' - (K - E)K' = \frac{1}{2}\pi$. *Proceed. London Math. Soc.*, (2) iii. pp. 206–224.]

So far back as 1839 Catalan was led by consideration of the theorem for the change of variables in a multiple integral to

* *Transac. R. Soc. Edinburgh*, xxxix. pp. 324–325.

arrive at the value of a determinant having definite integrals for elements, his first result being seen to produce Legendre's theorem on taking the said determinant to be of the third order (*Hist.*, iii. pp. 469–473). The title of the paper which we have now reached suggests that history may be repeating itself, and after a fashion such is found to be the case. In content, however, the two papers bear little resemblance, the latter being by far the more comprehensive, standing alone indeed as a storehouse of evaluations of the Catalan type of determinant, besides being of value otherwise.

The first result obtained is that *If we put*

$$H(a_s, a_{s+1}; m) \quad \textit{for} \quad \int_{a_{s+1}}^{a_s} \prod_{t=1}^{t=n} (x - a_t)^{\beta_t - 1} x^m dx$$

then

$$\begin{vmatrix} H(a_1, a_2, n-2) & H(a_1, a_2, n-3) & \dots H(a_1, a_2, 1) & H(a_1, a_2, 0) \\ H(a_2, a_3, n-2) & H(a_2, a_3, n-3) & \dots H(a_2, a_3, 1) & H(a_2, a_3, 0) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ H(a_{n-1}, a_n, n-2) & H(a_{n-1}, a_n, n-3) & \dots H(a_{n-1}, a_n, 1) & H(a_{n-1}, a_n, 0) \end{vmatrix}$$

$$= \Pi\{(-1)^{n-1} f^1(a_s)\}^{\beta_s - 1} \cdot \Pi(a_s - a_t) \frac{\Pi\Gamma(\beta_s)}{\Gamma(\Sigma\beta_s)},$$

where $\qquad f(x) = (x - a_1)(x - a_2) \dots (x - a_n),$

and s *and* t *take all the values* 1, 2, . . . , n *subject to the condition* s > t: and two specializations are

$$\begin{vmatrix} \int_{a_2}^{a_1} x^3 R^{-\frac{1}{2}} dx & \int_{a_2}^{a_1} x^2 R^{-\frac{1}{2}} dx & \int_{a_2}^{a_1} x R^{-\frac{1}{2}} dx & \int_{a_2}^{a_1} R^{-\frac{1}{2}} dx \\ \int_{a_3}^{a_2} x^3 R^{-\frac{1}{2}} dx & \int_{a_3}^{a_2} x^2 R^{-\frac{1}{2}} dx & \int_{a_3}^{a_2} x R^{-\frac{1}{2}} dx & \int_{a_3}^{a_2} R^{-\frac{1}{2}} dx \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \int_{a_5}^{a_4} x^3 R^{-\frac{1}{2}} dx & \int_{a_5}^{a_4} x^2 R^{-\frac{1}{2}} dx & \int_{a_5}^{a_4} x R^{-\frac{1}{2}} dx & \int_{a_5}^{a_4} R^{-\frac{1}{2}} dx \end{vmatrix} = -\tfrac{4}{3}\pi^2,$$

$$\begin{vmatrix} \int_{a_2}^{a_1} x Q^{-\frac{1}{2}} dx & \int_{a_2}^{a_1} Q^{-\frac{1}{2}} dx \\ \int_{a_3}^{a_2} x Q^{-\frac{1}{2}} dx & \int_{a_3}^{a_2} Q^{-\frac{1}{2}} dx \end{vmatrix} = 2\pi i,$$

where $R = (x - a_1)(x - a_2) \ldots (x - a_5)$ and

$$Q = (x - a_1)(x - a_2)(x - a_3).$$

Among other matters following on this is the specially interesting theorem that the ratio of an H determinant of the $(n - r + 1)^{\text{th}}$ order to a similar one of the r^{th} order can be expressed as a product of gamma functions. Finally it is shown that to these quite a series of analogues may be obtained by taking the different basic integral

$$\int e^{-x} \prod_{s=1}^{s=n} (x - a_s)^{\beta_s - 1} dx$$

instead of H.

NESBITT, A. M. ($1904^1/_{10}$)

[Problem 15649. *Educ. Times*, lvii. p. 449.]

The result here announced but unproved is that *the* n-*line determinant whose* (r, s)th *element is* $(n+1)_{2s-r}$ *is equal to* $2^{\frac{1}{2}n(n+1)}$.

We note for ourselves that the determinant is centrosymmetric, and that its centrosymmetry leads to evaluation. For example, when n is 4 the determinant is $|(5)_{2s-r}|$

$$\text{i.e.} \begin{vmatrix} 5_1 & 5_3 & 5_5 & \cdot \\ 5_0 & 5_2 & 5_4 & \cdot \\ \cdot & 5_1 & 5_3 & 5_5 \\ \cdot & 5_0 & 5_2 & 5_4 \end{vmatrix} \quad \text{i.e.} \begin{vmatrix} 5 & 10 & 1 & \cdot \\ 1 & 10 & 5 & \cdot \\ \cdot & 5 & 10 & 1 \\ \cdot & 1 & 10 & 5 \end{vmatrix}$$

$$\text{which} = \begin{vmatrix} 5 & 10+1 \\ 1 & 10+5 \end{vmatrix} \cdot \begin{vmatrix} 5 & 10-1 \\ 1 & 10-5 \end{vmatrix} = 64 \cdot 16 = 2^{10}.$$

BAKER, R. P. (1904/$_{12}$)

[The expression of the areas of polygons in determinant form. *American Math. Monthly*, xi. pp. 227–228.]

The subject really is the inexpressibility of the area of an n-sided polygon by means of an n-line determinant whose first three columns are

$$\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \cdot & \cdot & \cdot \\ x_n & y_n & 1. \end{matrix}$$

THIELE, T. N. ($1905^{12}/_1$): NÖRLUND, N. E. ($1908^{21}/_8$), ($1910^{10}/_1$)

[Différences réciproques. *Oversigt... Forhandlinger* (København), 1906, pp. 153–171.]

[Sur les différences réciproques. *Comptes rendus . . . Acad. des Sci.* (Paris), cxlvii. pp. 521–524.]

[Fonctions continues et différences réciproques. *Acta Math.*, xxxiv. pp. 1–105.]

It would appear that the functions, named 'reciprocal differences' by Thiele, came up first before him as separate realities of potential value when he was engaged on a searching investigation of the subject of interpolation. The name indeed carries with it an indication of the possible truth of such a supposition, the new 'differences' being closely related to the '*differentias divisas*' of Newton as used by him in the solution of his interpolation-problem (*Principia*, lib. iii., lemma 5). As often happens, the general name conveys very little of the definite something in the user's mind: besides, it is the unabridged expression 'the reciprocal differences of a set of arguments with respect to a function' that requires explanation. This is given by the equations

$$R(a) = \phi(a), \quad R(a, b) = \frac{b-a}{\phi(b) - \phi(a)},$$

$$R(a,b,c,\ldots,f,g) = \frac{g-a}{R(b,c,\ldots,f,g) - R(a,b,\ldots,f)} + R(b,c,\ldots,f),$$

where R stands for reciprocal difference, $a, b, c, \ldots$ are the arguments in question, and ϕ is the function. Our first use of them obtains readily for us

$$R(a, b, c) = \begin{vmatrix} 1 & \phi(a) & a\phi(a) \\ 1 & \phi(b) & b\phi(b) \\ 1 & \phi(c) & c\phi(c) \end{vmatrix} \div \begin{vmatrix} 1 & \phi(a) & a \\ 1 & \phi(b) & b \\ 1 & \phi(c) & c \end{vmatrix};$$

then with a certain increase of trouble

$$R(a, b, c, d) = \begin{vmatrix} 1 & \phi(a) & a & a^2 \\ 1 & \phi(b) & b & b^2 \\ 1 & \phi(c) & c & c^2 \\ 1 & \phi(d) & d & d^2 \end{vmatrix} \div \begin{vmatrix} 1 & \phi(a) & a & a\phi(a) \\ 1 & \phi(b) & b & b\phi(b) \\ 1 & \phi(c) & c & c\phi(c) \\ 1 & \phi(d) & d & d\phi(d) \end{vmatrix},$$

and so on. Of the two types of alternants which thus present themselves one is a little more familiar than the other, but, of course, both are included in Garbieri's general theorem of 1878 (*Hist.*, iii. pp. 163–165: iv. pp. 155–157). It is the quotient that is really new, the nearest approach to it hitherto being the case of it where the divisor is the difference-product of the variables. It is just worth adding that since both dividend and divisor are divisible by the said product we may look upon R as being the quotient of two bi-alternant functions (see above, pp. 187–190).

As regards the use to which the new differences are put (§§ 2, 3) it must suffice to indicate briefly how the main result is obtained in instalments by repeating the use of the recurrence-formula, thus:—

$$\phi(x) = \phi(d) + \frac{x-d}{R(d,x)}, \quad = \phi(d) + \cfrac{x-d}{R(c,d) + \cfrac{x-c}{R(c,d,x) - \phi(d)}}$$

$$= \phi(d) + \cfrac{x-d}{R(c,d) + \cfrac{x-c}{R(b,c,d) - \phi(d) + \cfrac{x-b}{R(b,c,d,x) - R(c,d)}}}$$

$$= \cdots\cdots\cdots\cdots\cdots\cdots$$

Finally, the author gives attention (§§ 4, 5) to the case in which all the arguments converge on one, and where, of course, differentiation has to be resorted to.

It would seem that Nörlund must have been early attracted to Thiele's publication, for his own comprehensive monograph on like subjects, although printed only in 1910, is dated at the end by its author $1907/_9$. It consists of five chapters, the fourth (pp. 55–89) being the one that bears the title 'différences réciproques'; and it is the early part of this that deals with the representation by determinants.

FITE, W. B. $(1906/_3)$

[Certain factors of the group determinant. *American Math. Monthly*, xiii. pp. 51–53.]

Determinants of the type here exemplified are something of a rediscovery, and we have already pointed out in the chapter on Circulants under the year $1902/_1$ that they can be changed

into centrosymmetric form, and resolved into factors accordingly. As a variant on this, however, we may now add that they can also be brought under the following general result: *If* (A), (B) *be the matrices of any two determinants of the* nth *order, the* 2n-*line determinant*

$$\begin{vmatrix} (A) & (B) \\ (B) & (A) \end{vmatrix} = |(A) + (B)| \cdot |(A) - (B)|.$$

It is in fact a block centrosymmetric, and thus to be associated with block circulants and other like specialties.

TAYLOR, W. W. (1906¼)

[Question 15967. *Educ. Times*, lix. pp. 193, 441.]

Virtually the first column of the very interesting determinant considered here is

$$\begin{gathered} (m+n)(n+l)(l+m) \\ la^2+mb^2+nc^2 \\ ld^2+me^2+nf^2 \\ (m-n)(n-l)(l-m)\cdot(la^2+mb^2+nc^2)(ld^2+me^2+nf^2), \end{gathered}$$

the second column being got from the first by changing the sign of l, the third by changing the sign of m, and the fourth by changing the sign of n: and the result of evaluation is

$$-16l^2m^2n^2\cdot\begin{vmatrix} \Delta(a,b,c) & \Delta(la,mb,nc) \\ \Delta(d,e,f) & \Delta(ld,me,nf) \end{vmatrix}$$

where

$$\Delta(a,b,c) = a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2.$$

An alternative mode of proof would be helpful.

What is essential in the matter, however, must not be obscured by bringing in a 4-line determinant, either the author's or ours. The real problem at the basis of all is to establish the equality of the 2-line determinant specified and the following 3-line determinant:

$$\begin{vmatrix} a^2 & d^2 & (m^2+n^2)D+2(m^2-n^2)(-l^2a^2d^2+m^2b^2e^2+n^2c^2f^2) \\ b^2 & e^2 & (n^2+l^2)D+2(n^2-l^2)(l^2a^2d^2-m^2b^2e^2+n^2c^2f^2) \\ c^2 & f^2 & (l^2+m^2)D+2(l^2-m^2)(l^2a^2d^2+m^2b^2e^2-n^2c^2f^2) \end{vmatrix}$$

where
$$D = \begin{vmatrix} 1 & l^2 & b^2f^2+c^2e^2 \\ 1 & m^2 & c^2d^2+a^2f^2 \\ 1 & n^2 & a^2e^2+b^2d^2 \end{vmatrix}.$$

They are both symmetrical with respect to the triad of cyclical substitutions,

$$l, m, n = m, n, l; \quad a, b, c = b, c, a; \quad d, e, f = e, f, d.$$

What creates difficulty is the fact that in the 3-line determinant the cyclo-symmetry is due to the rows constituting a cycle, whereas in the 2-line determinant it is due to the elements themselves being individually cyclo-symmetric.

MUIR, T. (1906³/₉)

[The sum of the r-line minors of the square of a determinant. *Proceed. R. Soc. Edinburgh*, xxvi. pp. 533–539.]

The special determinants incidentally drawn attention to here (§ 7) are those in which the elements are oblong arrays: for example,

$$\begin{vmatrix} \begin{Vmatrix} b_1 & b_2 & b_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Vmatrix} & \begin{Vmatrix} b_1 & b_2 & b_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{Vmatrix} \\ \\ \begin{Vmatrix} c_1 & c_2 & c_3 \\ \beta_1 & \beta_2 & \beta_3 \end{Vmatrix} & \begin{Vmatrix} c_1 & c_2 & c_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{Vmatrix} \end{vmatrix} = \begin{Vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} \cdot \begin{Vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{Vmatrix}.$$

More noteworthy is a series of determinants in which only the non-diagonal elements are arrays: for example,

$$\begin{vmatrix} a_1+a_2+\dots+a_m & (a_1, a_2, \dots, a_m) \\ (b_1, b_2, \dots, b_m) & b_1+b_2+\dots+b_m \end{vmatrix} = \Sigma \overset{+}{|} a_1b_2 \overset{+}{|},$$

$$\begin{vmatrix} a_1+a_2+\dots+a_m & (a_1, a_2, \dots, a_m) & (a_1, a_2, \dots, a_m) \\ (b_1, b_2, \dots, b_m) & b_1+b_2+\dots+b_m & (b_1, b_2, \dots, b_m) \\ (c_1, c_2, \dots, c_m) & (c_1, c_2, \dots, c_m) & c_1+c_2+\dots+c_m \end{vmatrix} = \Sigma \overset{+}{|} a_1b_2c_3 \overset{+}{|}.$$

To these last equalities a peculiar interest attaches, the right-hand members being expressions that date back to Waring (1762) and Binet (1811, *Hist.*, i. p. 81), and that had been sought

to be condensed by Brioschi (1854, *Hist.*, iii. pp. 473–474), by Bellavitis (1857, *Hist.*, ii. p. 96), and by Bruno in a thesis of 1856 referred to on p. 10 of his ‘ Théorie des Formes Binaires ’.

It need hardly be pointed out that the determinants of this paper have no real resemblance to block determinants, the constituent arrays in the two cases being quite unlike in character and function. Here they are non-quantitative elements, such that the so-called product of two of them is quantitative. In the block determinant on the other hand they are not themselves elements, but are collections of the real elements which it is a convenience to have segregated by partitions.

KÜRSCHÁK, J. ($1906/_{10,\,11}$)

[Bizonyos determinánsok jellemző tulajdonságairól. *Math. és Phys. Lapok*, xv. pp. 270–276.]

The distinguishing traits of the new form here introduced may seem trifling : this however is not the character of the communication itself, which makes an interesting contribution to the subject of what we may call the defining characteristics of a determinant viewed as a function of its elements—a subject which we have last seen touched on by Hensel and Frobenius (Chap. I above). Kürschák's determinant is simply one in which there are two kinds of rows, rows of variables and rows of constants, for example:

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{m1} & x_{m2} & \cdots & x_{m,\,n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-m,\,1} & a_{n-m,\,2} & \cdots & a_{n-m,\,n} \end{vmatrix}, \text{ or } F(x_1, x_2, \ldots, x_m) \text{ say,}$$

the defining properties being (1) that it is linear and homogeneous in each of the variable rows, (2) that it is altered in sign when two such rows are interchanged, (3) that it is such that

$$F(x_1, \ldots, x_m) \cdot F(y_1, \ldots, y_\rho, \ldots, y_m) \\ = \sum_{\rho=1}^{m} F(y_\rho, y_2, \ldots, y_m) \cdot F(y_1, \ldots, x_1, \ldots, y_m)$$

—in other words, such that the product of two F's is expressible as a sum of like products after the manner exemplified in Sylvester's theorem of 1839 (*Hist.*, i. p. 233, ii. pp. 61–62).

The five-page proof is not so convincing as interesting.

NESBITT, A. M. ($1907^1/_1$)

[Question 16132. *Educ. Times*, lx. p. 37: or *Math. from Educ. Times*, (2) xii. pp. 59–60.]

The result established here, supplemented by another, is

$$\begin{vmatrix} \cdot & x\cos C+y & x\cos B+z \\ y\cos C+x & \cdot & y\cos A+z \\ z\cos B+x & z\cos A+y & \cdot \end{vmatrix}$$
$$= \begin{vmatrix} xyz+1 & y^2z-\cos C & yz^2-\cos B \\ zx^2-\cos C & xyz+1 & z^2x-\cos A \\ yx^2-\cos B & y^2x-\cos A & xyz+1 \end{vmatrix}$$
$$= \Sigma(x\sin A)\cdot\Sigma(yz\sin A),$$

where A, B, C are the angles of a triangle. We note for ourselves that the determinants are got by multiplying

$$\begin{vmatrix} 1 & -x\cos C & -x\sin C \\ 1 & y & \cdot \\ 1 & -z\cos A & z\sin A \end{vmatrix} \text{ and } \begin{vmatrix} yz & \cos C & \sin C \\ zx & -1 & \cdot \\ xy & \cos A & -\sin A \end{vmatrix}$$
$$\text{by } \begin{vmatrix} x & \cos C & \sin C \\ y & -1 & \cdot \\ z & \cos A & -\sin A \end{vmatrix}.$$

DICK, G. R. ($1906^1/_{10}$, $1907^1/_6$)

[Questions 16115, 16220. *Educ. Times*, lix. p. 540; lx. p. 270; or *Math. from Educ. Times*, (2) xii. pp. 53–54; xiii. pp. 45–46.]

The second result

$$\begin{vmatrix} a_1+a_2-a_4-a_5 & a_1a_2-a_4a_5 & a_1a_2(a_4+a_5)-a_4a_5(a_1+a_2) \\ a_2+a_3-a_5-a_6 & a_2a_3-a_5a_6 & a_2a_3(a_5+a_6)-a_5a_6(a_2+a_3) \\ a_3+a_4-a_6-a_1 & a_3a_4-a_6a_1 & a_3a_4(a_6+a_1)-a_3a_1(a_3+a_4) \end{vmatrix} = 0$$

is here established by performing the operation

$$\text{row}_1\cdot(a_3-a_6)\;+\;\text{row}_2\cdot(a_4-a_1)\;+\;\text{row}_3\cdot(a_5-a_2).$$

The first is of less interest.

'ANON.' (1907)

[Question 17041. *Educ. Times*, lxiv. (1911), p. 137: or *Math. from Educ. Times*, (2) xxi. pp. 28–30.]

Two proofs are here given of a result dating from the Cambridge mathematical tripos of 1907, namely: *If*

$$\nabla^2\phi = \nabla^2\psi = \nabla^2\chi = 0,$$

and we denote

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{vmatrix}, \quad \begin{vmatrix} \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{vmatrix}, \quad \begin{vmatrix} \dfrac{\partial\chi}{\partial x} & \dfrac{\partial\chi}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{vmatrix} \quad \textit{by } P,\ Q,\ R$$

then

$$\begin{vmatrix} P & P & P \\ Q & Q & Q \\ R & R & R \end{vmatrix} \quad \text{i.e. } PQR - PRQ + \ldots - RQP$$

$$= \begin{vmatrix} \dfrac{\partial^2\phi}{\partial x\partial y} & \dfrac{\partial^2\psi}{\partial x\partial y} & \dfrac{\partial^2\chi}{\partial x\partial y} \\ \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\psi}{\partial y} & \dfrac{\partial\chi}{\partial y} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\chi}{\partial x} \end{vmatrix} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

MUIR, T. ($1907^1/_9$)

[Brioschi's $2m$-line determinant with elements subject to $m(2m-1)$ conditions. *Messenger of Math.*, xxxvii. pp. 107–111.]

The determinant here dealt with is that used by Brioschi (*Hist.*, ii. pp. 276–277) in his attempted proof that the product

of two sums of eight squares is itself expressible as a sum of eight squares, namely,

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ -b & a & -d & c & -f & e & -h & g \\ -c & d & a & -b & -g & h & e & -f \\ -d & -c & b & a & -h & -g & f & e \\ e & f & g & h & a & b & c & d \\ -f & e & -h & g & -b & a & -d & c \\ -g & h & e & -f & -c & d & a & -b \\ -h & -g & f & e & -d & -c & b & a \end{vmatrix}, \text{ or } B \text{ say.}$$

In opposition to Brioschi's assertion that B is equal to $(\Sigma a^2)^4$ it is here proved in two different ways to have the value $(\Sigma a^2 - M)^2 (\Sigma a^2 + M)^2$, where $M = 2(ae + bf + cg + dh)$, the simpler method turning on the multiplication of B row-wise by itself, and so obtaining

$$\begin{vmatrix} \Sigma a^2 & . & . & . & M & . & . & . \\ . & \Sigma a^2 & . & . & . & M & . & . \\ . & . & \Sigma a^2 & . & . & . & M & . \\ . & . & . & \Sigma a^2 & . & . & . & M \\ M & . & . & . & \Sigma a^2 & . & . & . \\ . & M & . & . & . & \Sigma a^2 & . & . \\ . & . & M & . & . & . & \Sigma a^2 & . \\ . & . & . & M & . & . & . & \Sigma a^2 \end{vmatrix}.$$

The bearings of the result are also commented on, Sylvester's determinant which really equals $(\Sigma a^2)^4$ being brought in (*Hist.*, iii. pp. 288–290).

MUIR, T. (1907⁷/₁₀)

[The product of the primary minors of an m-by-$(m+1)$ array. *Proceed. R. Soc. Edinburgh*, xxviii. pp. 210–216.]

The product in question is here expressed as a determinant of the order $\frac{1}{2}m(m+1)$. For example, if m be 3 and the array be

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4, \end{matrix}$$

the equality is

$$\begin{vmatrix} a_1a_2 & b_1b_2 & c_1c_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & b_1c_2+b_2c_1 \\ a_1a_3 & b_1b_3 & c_1c_3 & a_1b_3+a_3b_1 & a_1c_3+a_3c_1 & b_1c_3+b_3c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_3a_4 & b_3b_4 & c_3c_4 & a_3b_4+a_4b_3 & a_3c_4+a_4c_3 & b_3c_4+b_4c_3 \end{vmatrix}$$

$$= - \mid a_1b_2c_3 \mid \cdot \mid a_1b_2c_4 \mid \cdot \mid a_1b_3c_4 \mid \cdot \mid a_2b_3c_4 \mid ,$$

where, be it noted as a help towards specifying the law of construction, every two columns of the array go to form a row of the determinant: for example,

$$(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) = \text{sum of elements of 1st row.}$$

The proof consists in so transforming the determinant as to bring out the fact that $| a_1b_2c_3 |$ is a factor, and at the same time to make it evident that a similar procedure would suffice in the case of any of the other 3-line minors. The connecting sign-factor is shown to be $(-1)^{\frac{1}{2}m(m-1)}$.

In the paper itself the example actually used to illustrate the mode of proof is that where m is 4: and the author reports that equalities which are in effect the cases where m is 2 and m is 3 had been used without proof about 1878 by A. Scholtz and Hunyady.

TERRACINI, A. (1908 $^{20}/_{1}$)

[Nota su una classe di determinanti. *Giornale di Mat.*, xlvii. pp. 25–32.]

The determinants here considered are those in which the sum of each element and its conjugate is constant. The class may be viewed as being more comprehensive than that of the zero-axial skew determinant, for there the said constant though it exists is equal to 0. Any determinant of the wider class may be formed from one of the narrower by adding the basic constant to each element of the latter. If therefore the former determinant be denoted by T, the latter by S, and the basic constant by ω we have from Muir's lemma of 1902

$$T = S + \omega S'$$

where S' is the sum of the signed primary minors of S and in form is a zero-axial skew determinant of the next higher order.

It thus follows that if the order of S be odd $S = 0$ and $T = \omega S'$: whereas if S be of even order $S' = 0$ and $T = S$: and in this the main theorem of the paper is included, namely, *A Terracini determinant is the product of either* 1 *or* ω *by the square of a Pfaffian according as the order of the determinant is odd or even.* The author's mode of arriving at it is lengthier but equally effective. His other result is that *if* T_{sr}, T_{rs} *be conjugate primary minors of* T, *then* $(\mathrm{T}_{rs} + \mathrm{T}_{sr})^2 = 4\mathrm{T}_{rs}\mathrm{T}_{sr}$.

KLUGE, W. (1908)

[Besondere Systeme. Ein Beitrag zur Bestimmung von Determinanten. Sch.-Progr. 48 pp. Lissa i. P.]

This is an interesting and creditable school-program; the subject is those determinants that have no elements other than 1, 0, −1; and the treatment is not at all commonplace. The author's reading in his subject is not extensive: indeed his pamphlet looks like the outcome of a critical study of certain sections of Pascal's textbook, from one of which his title is borrowed. In more favourable circumstances his work would have been still more valuable.

The first half (pp. 7–27) is occupied with the re-evaluation of a longish series of familiar determinants of various types, not all of them restricted in their elements to 1, 0, −1, the most noteworthy result perhaps being the rediscovery and rectification of Catalan's mistake of 1846 in regard to the circulant $C(-1, -1, \ldots, -1, 1, 1, \ldots)$.

The second half (pp. 28–48) is devoted to that portion of Hadamard's paper of 1893 (*Hist.*, iv. pp. 483–484) which deals with the formation of determinants of maximum value, here called maximal determinants. Considerable success is attained, due largely to the skilful use made of Zehfuss' theorem of 1858 (*Hist.*, ii. pp. 102–104) regarding the formation of a determinant equal to P^qQ^p where P is a determinant of the p^{th} order and Q one of the q^{th} order.

DIXON, A. L. ($1908^{11}/_{6}$)

[On a form of the eliminant of two quantics. *Proceed. London Math. Soc.*, (2) vi. pp. 468–478.]

The form in question is Borchardt's of 1859 (*Hist.*, ii. pp.

147–150, 458–459), and naturally Sylvester's unisignant of 1855 comes up for consideration. A page or so is occupied (§ 5) with a fresh proof of the latter's result regarding the number of terms in it (*Hist.*, ii. pp. 406–407). Neither Sylvester nor Borchardt is referred to: but this the author rectifies in one of a series of subsequent papers in which he makes noteworthy advances on Borchardt's work.

MUIR, T. (1908[1]/9)

[Question 16494. *Educ. Times*, lxi. p. 412.]

The result announced here is that *if* $c_{rs} = a_{rs} + b_{rs}\sqrt{-1}$, $c_{sr} = a_{rs} - b_{rs}\sqrt{-1}$ *and* $c_{rr} = 0$, *then* $|\, c_{11}\, c_{22}\, c_{33}\, c_{44} \,|$ *is equal to the sum of two expressions of the form*

$$l^2+m^2+n^2-2mn-2nl-2lm+(p-q+r)^2+2\begin{vmatrix} l & p & 1 \\ m & -q & 1 \\ n & r & 1 \end{vmatrix}.$$

CORBIN, J. C. (1908/11)

[Question 306. *American Math. Monthly*, xv. p. 214: xvi. p. 31.]

Attention is here drawn to Muir's identity

$$\begin{vmatrix} 1 & a & a & a^2 \\ 1 & b & b & b^2 \\ 1 & c & c' & cc' \\ 1 & d & d' & dd' \end{vmatrix} = (b-a)\begin{vmatrix} 1 & a+b & ab \\ 1 & c+d' & cd' \\ 1 & c'+d & c'd \end{vmatrix},$$

a proof of it being given. Its connection, however, with a corresponding result of Cayley's of the year 1858 (*Hist.*, ii. p. 453) is not noted, and there are thus missed the interesting deductions

$$\begin{vmatrix} 1 & a & b & ab \\ 1 & b & a & ba \\ 1 & c & d & cd \\ 1 & e & f & ef \end{vmatrix} = \begin{vmatrix} 1 & a & a & a^2 \\ 1 & b & b & b^2 \\ 1 & c & e & ce \\ 1 & f & d & fd \end{vmatrix},$$

$$\begin{vmatrix} 1 & a+b & ab \\ 1 & c+d & cd \\ 1 & e+f & ef \end{vmatrix} = \begin{vmatrix} 1 & 1 & a+b \\ c & e & ce+ab \\ f & d & fd+ab \end{vmatrix}, \qquad \begin{vmatrix} c+d & cd \\ e+f & ef \end{vmatrix} = \begin{vmatrix} d & e & e \\ d & . & c \\ f & f & c \end{vmatrix}.$$

BURKHARDT, H. AND VOGT, H. ($1909^{17}/_{8}$)

[Caractères d'un groupe quelconque. Déterminant d'un groupe. *Encycl. des Sci. Math.*, i. (1), pp. 613–616.]

Nothing of note as regards the determinant.

GAMBERINI, G. ($1908/_{10}$, 1909)

[Una speciale classe di matrici quadrate permutabili. *Giornale di Mat.*, xlvii. pp. 137–155.]

[Alcuni resultati intorno al prodotto di due matrici quadrate. 8 pp. Roma.]

[Sulle relazioni fra alcune forme del prodotto di due matrici quadrate di ordine n. Pp. 9-12 of preceding.]

The term 'permutable' applied to square arrays is used here as in the theory of linear substitutions—that is to say, two square arrays are said to be permutable when their row-by-column product is independent of the order in which the two are taken: thus the n^2 conditions for $|a_{1n}|$ and $|b_{1n}|$ being permutable are

$$\sum_{k=1}^{k=n} a_{rk}b_{ks} = \sum_{k=1}^{k=n} b_{rk}a_{ks} \qquad (r, s = 1, 2, \ldots, n).$$

In the first paper the general subject of such arrays is entered on, but is soon diverged from and gives place to that of magic squares, which, strange to say, seems to be the main matter of interest to the author. The second paper is a continuation of the first and like the third was probably meant to be published in the same serial as the first. This did not take place, however, and the two appear to have been privately printed.*

KOWALEWSKI, G. (1909)

[Einführung in die Determinantentheorie. Kap. xv. pp. 320–337.]

The determinant here brought into notice by association with

* In the second paper the title of the first is misquoted '*Su di* una classe *notevole* . . .', and some little misunderstanding has thence arisen.

the Wronskian has definite integrals for its elements and is axisymmetric in form, its $(r, s)^{\text{th}}$ element being

$$\int_a^b \phi_r \phi_s \, dx.$$

The author calls it Gram's determinant, and reference is made by him to a paper of J. F. Gram's of the year 1881 (*Hist.*, iv. p. 457). This paper, however, makes no mention of the determinant in question, and thus far we have not come across any other paper by the same writer that is more explicit. It is evident that if a homogeneous linear relation connect the ϕ's the determinant vanishes: for

if $$\lambda\phi_1 + \mu\phi_2 + \nu\phi_3 = 0,$$

then $$\lambda \, \text{row}_1 + \mu \, \text{row}_2 + \nu \, \text{row}_3 = 0.$$

But in addition to this it is proved that if the ϕ's be linearly independent, the determinant is positive. A test is thus established for the linear dependence of a set of functions (1) when the functions are real and continuous, (2) when they are complex continuous functions of one variable.

As regards the naming of the determinant it is not unimportant to add that the author also calls the square of an m-by-n array "the Gramian determinant of the array".

NICOLETTI, O. (1909²/₅)

[Sulla caratteristica del determinante di una forma di Hermite. *Rendic. . . . Accad. dei Lincei*, (5) xviii. pp. 428–431.]

The reply given to the question implied in the title here is: *The necessary and sufficient condition that the Hermitant* $|\mathrm{a}_{hk}|_n$ *shall have the characteristic* (*or rank*) r *is that*

$$\text{saxm}_{r+2} = \text{saxm}_{r+1} = 0, \ \ \text{saxm}_r \neq 0,$$

where saxm_r *stands for* '*sum of* r-*line coaxial minors*'. It is reached by establishing first a more general proposition involving a second Hermitant $|b_{hk}|_n$, and then specializing the latter. This auxiliary Hermitant is only less general than the other in being such that its related quadric $\Sigma\, b_{hk} x_h \bar{x}_k$ is not indefinite: and the condition in question is found in the form

$$C_{r+2} = C_{r+1} = 0, \quad C_r \neq 0,$$

where C_r is the coefficient of $\omega_1{}^r\omega_2{}^{n-r}$ in the expansion of

$$| a_{hk}\omega_1 + b_{hk}\omega_2 |.$$

The requisite specialization referred to is $b_{hh} = 1$, $b_{hk} = 0$. Of the other specializations discussed the most interesting are those which lead to results regarding two forms of determinants with *real* elements, namely, axisymmetric and zero-axial skew,—determinants first thus viewed in connection with their rank by Frobenius in 1876 (*Hist.*, iii. pp. 275–276).

SANJANA, K. J. (1909/8)

[A determinant theorem and its geometrical application. *Indian Math. Club*, i. pp. 134–136.]

We may express the theorem as follows: *If every row*

$$r1 \quad r2 \; . \; . \; . \; rn$$

of a determinant be such that r1 *is* 1, r2 *is* a_1 *or* a_2, r3 *is* b_1 *or* b_2, r4 *is* c_1 *or* c_2, *and so on, save that* rn *is the sum of the* k^{th} *powers of* r1, r2, r3, . . . , *then the determinant vanishes.* It is readily proved by subtracting the first row from each of the others, and reducing the last column of the $(n-1)$-line determinant thus got to a column of zeros.

LANDSBERG, G. (1909/8)

[Theorie der Elementartheiler linearer Integralgleichungen. *Math. Annalen*, lxix. pp. 227–265.]

Of this important paper only a short section (pp. 231–232) bearing the title 'Das Multiplicationstheorem für Integraldeterminanten' falls to be noted here. What it contains is a formal enunciation and proof of the equality

$$\left| \int_a^b \phi_r \psi_s dx \right|_n = \frac{1}{n!} \int_a^b \int_a^b \ldots | \phi_r(x_s) |_n \, | \psi_r(x_s) | \, dx_1 dx_2 \ldots dx_n$$

as given originally by Andréieff in 1883 (*Hist.*, iv. p. 464).

RICHARDSON, R. G. D. AND HURWITZ, W. A. (1909 14/9)

[Note on determinants whose terms are certain integrals. *Bull. American Math. Soc.*, (2) xvi. pp. 14-19.]

By 'terms' in the title are meant what are usually called 'elements', and the integrals referred to are of the form

$$\int_R \phi_r \psi_s dR$$

where the ϕ's and the ψ's denote functions which are limited and integrable in a certain region R of m-dimensional space. If there be put

D to stand for $\left| \int_R \phi_r \psi_s dR \right|_n,$

D_i „ „ D with its i^{th} row replaced by $\psi_1, \ldots, \psi_n$,

$\overline{D_j}$ „ „ D with its j^{th} column replaced by $\phi_1, \ldots, \phi_n$,

Δ_{ij} „ „ the cofactor of the $(ij)^{\text{th}}$ element of D,

the first of the author's two theorems is

$$\int D_i \overline{D}_j dR = D\Delta_{ij}.$$

Interest is considerably increased by drawing attention to certain special cases already more or less known, and spoken of here as the inequalities of Schwarz and Bessel. It is to be regretted, however, that in connection with the second theorem the authors, like Landsberg, make no reference to Andréieff's early acquaintance with it.

BRUN, F. DE (1910 1/6, 7/12)

[Un théorème sur les déterminants. *Arkiv för Mat.* . . . , vi. No. 25, 3 pp.]

[Sur une extension à l'espace à n dimensions du théorème du cosinus. *Arkiv för Mat.* . . . , vi. No. 37, 5 pp.]

The theorem of the first paper here is really a theorem regarding the special determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_{11} & x_{12} & x_{13} & \ldots & x_{1,n+1} \\ x_{21} & x_{22} & x_{23} & \ldots & x_{2,n+1} \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ x_{n1} & x_{n2} & x_{n3} & \ldots & x_{n,n+1} \end{vmatrix}, \text{ or } D \text{ say,}$$

where

$$\sum_{i=1}^{i=n} (x_{ij} - x_{ik})^2 = a_j^2 + a_k^2 \qquad \left(\begin{matrix} j = 1, 2, \ldots, n+1 \\ k = 1, 2, \ldots, n+1 \end{matrix} \quad j \neq k\right)$$

and is simply that

$$D^2 = a_1^2 a_2^2 \ldots a_n^2 \cdot \sum_{i=1}^{i=n} \frac{1}{a_i^2}.$$

The interest of it is chiefly geometrical, it being viewable as the generalization of the relation connecting the sides of a right-angled triangle.

The second paper is a natural extension of the first, the sum of the squares of the differences of the x's being now

$$\text{not } a_j^2 + a_k^2 \text{ but } a_j^2 + a_k^2 - 2a_j a_k \cos \alpha_{jk}.$$

The two take us back to the 'mutual-distance' theorems of 1841 (*Hist.*, ii. pp. 7–8, 109–110, 122–123, &c.).

MUIR, T. (1910 13/6)

[The less common special forms of determinants up to 1860. *Proceed. R. Soc. Edinburgh*, xxxi. pp. 311–332.]

About two dozen writings are here reported on. Four of the fresh forms which they deal with seem, from the attention given to them in the short period, to be already almost worthy of chapter-headings to themselves. Indeed since the close of the period three of the four have had as a consequence special names attached to them—permanents, Hermitants, unisignants,—the single one whose fate has not corresponded with prophecy being that used by Cayley in connection with work on anharmonic ratios (*Hist.*, ii. pp. 189, 451–6).

MUIR, T. ($1911^{15}/_3$)

[Sylvester's and other unisignants. *Transac. R. Soc. South Africa*, ii. pp. 187–195.]

After an introduction telling of Sylvester's unisignant S_n and its properties (*Hist.*, ii. pp. 456–457) the author points out that *the determinant*, T_n *say, obtained from it by changing the signs of all the non-diagonal elements is also free of negative terms in its final development.* In discussing this it is shown that in T_4, i.e.

$$\begin{vmatrix} a+a_2+a_3+a_4 & -a_2 & -a_3 & -a_4 \\ -b_1 & b_1+b+b_3+b_4 & -b_3 & -b_4 \\ -c_1 & -c_2 & c_1+c_2+c+c_4 & -c_4 \\ -d_1 & -d_2 & -d_3 & d_1+d_2+d_3+d \end{vmatrix}$$

the cofactors of *abcd*, *abc*, *abd*, *acd*, *bcd*, *ab*, *ac*, *ad*, *bc*, *bd*, *cd* are identical with the corresponding cofactors in S_4: that the cofactors of a, b, c, d are greater by

$$2(b_3c_4d_2 + b_4c_2d_3),\ 2(a_3c_4d_1 + a_4c_1d_3),$$
$$2(a_2b_4d_1 + a_4b_1d_2),\ 2(a_2b_3c_1 + a_3b_1c_2);$$

but that peculiar interest attaches to the aggregate of terms that are free of a, b, c, d, namely,

$$\begin{vmatrix} a_{234} & a_2 & a_3 & a_4 \\ b_1 & b_{134} & b_3 & b_4 \\ c_1 & c_2 & c_{124} & c_4 \\ d_1 & d_2 & d_3 & d_{123} \end{vmatrix}, \text{ or } 2\mathfrak{T} \text{ say,}$$

where for shortness' sake a_{234} is written for $a_2 + a_3 + a_4$, and where $\mathfrak{T}_4$ is less general than the unisignant,

$$\begin{vmatrix} a_{234} & a_{34} & a_{24} & a_{23} \\ -b_{134} & b_0 & -b_{14} & -b_{13} \\ -c_{124} & -c_{14} & c_0 & -c_{12} \\ -d_{123} & -d_{13} & -d_{12} & d_0 \end{vmatrix}, \text{ or } \mathfrak{T}' \text{ say.}$$

The study of the triad of new determinant forms thus led up to

closes with the theorem that *the numbers of terms in* $\mathfrak{T}_n$, T_n, $\mathfrak{T}'_n$ *are respectively*

$$(n-1)(n-2)^{n-1}, \qquad (2n-1)(n-1)^{n-1}, \qquad (n-1)^n.$$

Returning then to Sylvester's unisignant the author notes the curious proposition that *if it*, S_n, *be bordered horizontally by* 0, 1, 1, 1, . . . *and vertically by* 0, —1, —1, —1, . . . *the resulting determinant is also unisignant.* The like property, he ascertains, does not hold for T_n, but holds for the axisymmetric unisignants B_n and M_n dealt with in Chapter IV. The case of M_n is the most interesting, as *the bordered unisignant is then transformable into a like unisignant of the next lower order*, the exact result for the third order being that *the sum of the signed primary minors of* $M(a_1;\ b_1, b_2, b_3;\ c_1, c_2, c_3)$ *is* $M(b_1 + c_1, b_2 + c_2, b_3 + c_3)$. It is noted also that the number of terms in the final development of the bordered M_n is $n \cdot 2^{(n-2)(n-1)}$.

SPUNAR, V. M. (1911/$_{6,\,7}$)

[Question 355. *American Math. Monthly*, xviii. pp. 135–136.]

Here there is given a solution of a set of non-homogeneous linear equations whose determinant is the circulant

$$C(1, 1, 1, 0, 0, \ldots, 0)_n.$$

What requires attention of course is the special form of determinant appearing as the numerator of each of the unknowns.

MACMAHON, P. A. (1911^{13}/$_6$)

[Memoir on the theory of the partition of numbers. Part VI. *Philos. Transac. R. Soc. London*, ccxi. pp. 345–373.]

Two hitherto unnoted determinants turn up together here (pp. 350–360) as numerator and denominator of an important function. The denominator is the simpler, namely, the persymmetric:

$$(-1)^{\frac{1}{2}n(n-1)} \cdot P(x^{\frac{1}{2}n(n-1)}, x^{\frac{1}{2}(n-1)(n-2)}, \ldots, x, 1, 1, x, \ldots, x^{\frac{1}{2}(n-1)(n-2)})$$

which without difficulty can be shown equal to

$$(1 - x)^{n-1}(1 - x^2)^{n-2} \ldots (1 - x^{n-2})^2(1 - x^{n-1})^1;$$

for example,

$$\begin{vmatrix} 1 & 1 & x \\ x & 1 & 1 \\ x^3 & x & 1 \end{vmatrix} = (1 - x)^2(1 - x^2)^1.$$

The general character of the numerator will be gathered from the first two cases

$$\begin{vmatrix} 1-x^{a+1} & 1-x^3 \\ x & 1 \end{vmatrix},$$

$$\begin{vmatrix} (1-x^{a+1})(1-x^{a+2}) & (1-x^b)(1-x^{b+1}) & x(1-x^{c-1})(1-x^c) \\ x(1-x^{a+2}) & 1-x^{b+1} & 1-x^c \\ x^3 & x & 1 \end{vmatrix},$$

where, it may be worth noting, the removal of the binomial factors from the elements of the numerator gives us the corresponding denominator.

SATYANARAYANA, M. (1912$^1/_2$)

[Question 17239. *Educ. Times*, lxv. p. 81.]

The determinant here set for evaluation is in effect one in which the elements of the first row are

$$\sin(u - x - y - z), \quad \cos(u - x - y - z),$$
$$\sin(u - x)\sin(u - y)\sin(u - z) + \cos u \sin(2u - x - y - z)$$

while the second and third rows are got by changing u, x, y, z into x, y, z, u.

MUIR, T. (1912$^1/_3$)

[Question 6001. *Educ. Times*, lxv. p. 139: or *Math. from Educ. Times*, (2) xxii. pp. 49–50.]

The subject here is the expression of the coefficients in the final expansion of a product of n linear functions of $x_1, x_2, \ldots, x_n$ by

means of permanents. The general form of the expansion will be gathered from the case where n is 4, namely,

$$
\begin{aligned}
&(a_1x+b_1y+c_1z+d_1w)(a_2x+\ldots+d_2w)(a_3x+\ldots)(a_4x+\ldots)\\
&= \frac{1}{4!}\overset{+}{\Big|}\, a_1a_2a_3a_4 \,\overset{+}{\Big|}\, x^4 + \ldots + \frac{1}{4!}\overset{+}{\Big|}\, d_1d_2d_3d_4 \,\overset{+}{\Big|}\, w^4\\
&+ \frac{1}{3!}\overset{+}{\Big|}\, a_1a_2a_3b_4 \,\overset{+}{\Big|}\, x^3y + \ldots + \frac{1}{3!}\overset{+}{\Big|}\, c_1d_2d_3d_4 \,\overset{+}{\Big|}\, zw^3\\
&+ \frac{1}{2!2!}\overset{+}{\Big|}\, a_1a_2b_3b_4 \,\overset{+}{\Big|}\, x^2y^2 + \ldots + \frac{1}{2!2!}\overset{+}{\Big|}\, c_1c_2d_3d_4 \,\overset{+}{\Big|}\, z^2w^2\\
&+ \frac{1}{2!}\overset{+}{\Big|}\, a_1a_2b_3c_4 \,\overset{+}{\Big|}\, x^2yz + \ldots + \frac{1}{2!}\overset{+}{\Big|}\, b_1c_2d_3d_4 \,\overset{+}{\Big|}\, yzw^2\\
&+ \frac{1}{1!}\overset{+}{\Big|}\, a_1b_2c_3d_4 \,\overset{+}{\Big|}\, xyzw.
\end{aligned}
$$

NEUBERG, J. ($1912^1/_5$): NANSON, E. J. ($1912/_{10}$)

[Questions 17295, 17385. *Educ. Times*, lxv. pp. 218, 438: or *Math. from Educ. Times*, (2) xxiv. pp. 74–75; xxiii. pp. 105–106.]

The essential fact in Neuberg's communication is that

$$
\begin{vmatrix} a_1+xb_1 & b_1+yc_1 & c_1+zd_1 & d_1+wa_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_4+xb_4 & b_4+yc_4 & c_4+zd_4 & d_4+wa_4 \end{vmatrix} = |\, a_1b_2c_3d_4 \,|\,(1-xyzw):
$$

and Nanson's result is that the equality of px^2+qx+r, $p'x^2+q'x+r'$, $p''x^2+q''x+r''$ entails the equality

$$
\begin{vmatrix} p & q & 1 \\ p' & q' & 1 \\ p'' & q'' & 1 \end{vmatrix}
\begin{vmatrix} q & r & 1 \\ q' & r' & 1 \\ q'' & r'' & 1 \end{vmatrix}
= \begin{vmatrix} r & p & 1 \\ r' & p' & 1 \\ r'' & p'' & 1 \end{vmatrix}^2,
$$

which latter suitably suggests itself as the eliminant of

$$
\left.\begin{aligned}
px^3 + qx^2 + rx - y &= 0\\
p'x^3 + q'x^2 + r'x - y &= 0\\
p''x^3 + q''x^2 + r''x - y &= 0
\end{aligned}\right\}.
$$

SWAMINARAYAN, J. C. $(1911/_{10},\ 1912^{1}/_{7},\ ^{1}/_{8})$

[Question 323. *Journ. Indian Math. Soc.*, iii. p. 209: iv. pp. 226–227.]

[Questions 17328, 17349. *Educ. Times*, lxv. pp. 302, 308.]

What we are given here is (1) the evaluation of a determinant related to the well-known penesymmetric (*Hist.*, iii. p. 125) whose value is $2af \cdot bg \cdot ch \cdot (af + bg + ch)^3$; (2) an easy evaluation dependent on the simple equalities

$$\begin{vmatrix} a_1{}^2 & a_1b_1 & b_1{}^2 \\ a_2{}^2 & a_2b_2 & b_2{}^2 \\ a_3{}^2 & a_3b_3 & b_3{}^2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & a_2a_3b_1{}^2 \\ a_2 & b_2 & a_3a_1b_2{}^2 \\ a_3 & b_3 & a_1a_2b_3{}^2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & b_2b_3a_1{}^2 \\ a_2 & b_2 & b_3b_1a_2{}^2 \\ a_3 & b_3 & b_1b_2a_3{}^2 \end{vmatrix};$$

and (3) the curious result

$$\begin{vmatrix} x - \xi & |c_2d_3| & |a_2b_3| \\ y - \eta & |c_3d_1| & |a_3b_1| \\ z - \zeta & |c_1d_2| & |a_1b_2| \end{vmatrix} = |a_1b_2c_3d_4|$$

conditioned by the four equations

$$a_1x + a_2y + a_3z + a_4 = 0, \quad c_1\xi_1 + c_2\eta + c_3\zeta + c_4 = 0,$$
$$b_1x + b_2y + b_3z + b_4 = 0, \quad d_1\xi_1 + d_2\eta + d_3\zeta + d_4 = 0.$$

For ourselves we add that, if the last result be what the proposer intended, it will be found interesting and helpful to prove as a preliminary that

$$\begin{vmatrix} x & |c_2d_3| & |a_2b_3| \\ y & |c_3d_1| & |a_3b_1| \\ z & |c_1d_2| & |a_1b_2| \end{vmatrix}$$

$$= |a_1b_4|\,|c_2d_3| + |a_2b_4|\,|c_1d_3| + |a_3b_4|\,|c_1d_2| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & \cdot \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix};$$

and this is readily done if we begin by multiplying the left-hand member columnwise by

$$\begin{vmatrix} a_1 & b_1 & . \\ a_2 & b_2 & . \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Further, if we are right as to the determinant which suggested the first result, it is appropriate to note the three forms of the said determinant, namely,

$$\begin{vmatrix} (bg+ch)^2 & b^2f^2 & c^2f^2 \\ a^2g^2 & (ch+af)^2 & c^2g^2 \\ a^2h^2 & b^2h^2 & (af+bg)^2 \end{vmatrix},$$

$$\begin{vmatrix} (bg+ch)^2 & a^2f^2 & a^2f^2 \\ b^2g^2 & (ch+af)^2 & b^2g^2 \\ c^2h^2 & c^2h^2 & (af+bg)^2 \end{vmatrix}, \quad \begin{vmatrix} (bg+ch)^2 & abfg & cahf \\ abfg & (ch+af)^2 & bcgh \\ cahf & bcgh & (af+bg)^2 \end{vmatrix},$$

and to call attention to what happens when in them we put $a = b = c = 1$ or $f = g = h = 1$.

TENCA, L. ($1912^1/_9$): SOMMERVILLE, D. M. Y. ($1913^1/_1$)

[Armonizante di due forme determinanti. *Periodico di Mat.*, (3) x. p. 41.]

[Question 17438. *Educ. Times*, lxvi. p. 38.]

The first assertion here is that if the forms in question be $|x_{(r-1)n+s}|$ and $|u_{(r-1)n+s}|$, then the so-called* harmonizant is equal to $(n!)^2$: and the second is that if each row of a determinant be obtained from the row $a_1, b_1, a_2, b_2, \ldots$ by a transposition of one or more pairs of elements which bear the same suffix, then the determinant vanishes.

* By Battaglini in *Giorn. di Mat.*, ix. (1871) p. 195.

PÓLYA, G. (1913³/₁): MÉTROD, G. (1913¹/₇): BIEZENO, C. B. (1913)

[Aufgabe 425. *Archiv d. Math. u. Phys.*, (3) xx. p. 271.]

[Question 4247. *L'Intermédiaire des Math.*, xx. p. 148.]

[Vraagstuk 115. *Wiskundige Opgaven*, xi. p. 302.]

The proposition here set for proof by Pólya is that $|a_{rs}|_n$ can never be negative if $a_{rs} = -\int_a^b f_r f_s dx$, $a_{rr} = \int_a^b (S - f_r^2) dx$, and $S = f_1^2 + f_2^2 + \ldots + f_n^2$: the second recommends that $|a_{rs}|_n$, $|b_{rs}|_n$ be studied for cases where $a_{rs} \equiv b_{rs}$ (mod. m): and what is brought forward in the third is merely Dostor's so-called theorem of 1874 (*Hist.*, iii. p. 484.)

SCORZA, G. (1913¹³/₆)

[Sopra una certa classe di determinanti e sulla forme Hermitiane. *Giornale di Mat.*, li. pp. 335–342.]

As an introductory lemma there is here given Voigt's theorem of 1882 (*Hist.*, iv. p. 460), the determinant however being now written in Igel's block-skew form of 1894, which when abridged is

$$\begin{vmatrix} (A) & (-B) \\ (B) & (A) \end{vmatrix}$$

where (A) and (B) are m-line square arrays and $(-B)$ is got from (B) by changing the signs of all the elements. A fresh corollary is also given, namely, that *if* (A) *be axisymmetric and* (B) *be zero-axial skew, the determinant is the square of a rational integral function of its elements.* A new type of determinant is then taken up, whose elements are got in a peculiar way from the elements of an axisymmetric m-line determinant A and a zero-axial skew determinant B of like order. The law of its construction is perhaps best specified by saying that its $(2r-1)^{\text{th}}$ row gets its elements from the r^{th} row of A and the r^{th} row of B, those from A occupying the odd-numbered places and those from B the even-numbered:

and the r^{th} column of B, those from A occupying now the even-numbered places and those from B the odd-numbered. For example, if A, B be

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \qquad \begin{vmatrix} . & x & y \\ -x & . & z \\ -y & -z & . \end{vmatrix},$$

then the new determinant derived from them is

$$\begin{vmatrix} a & . & h & x & g & y \\ . & a & -x & h & -y & g \\ h & -x & b & . & f & z \\ x & h & . & b & -z & f \\ g & -y & f & -z & c & . \\ y & g & z & f & . & c \end{vmatrix}.$$

The usually attendant evaluation-theorem is included in a more comprehensive statement, namely, *If* Δ_r *denote the first coaxial minor of the* r*th order, then, when* r *is even* Δ_r *is an exact square, and when* r *is odd* $\Delta_r{}^2 = \Delta_{r-1}\Delta_{r+1}$. For example, in the 6-line instance just written

$$\Delta_2 = a^2, \quad \Delta_3 = a(ab - h^2 - x^2), \quad \Delta_4 = (ab - h^2 - x^2)^2, \quad \ldots$$

In regard to proof it is enough to suggest, as a first step, such a transposition of rows and of columns as will make applicable the corollary above noted.

MAILLET, E. (1913¹/₁₀)

[Question 4269. *L'Intermédiaire des Math.*, xx. p. 218: reply by E. Malo, xxi. pp. 173–176.]

The peculiar determinant, here for the first time spoken of, is one of the most interesting of our chapter. It may be defined as being

$$|a_{rs}|_{\frac{1}{2}(p-1)}$$

where p is a prime and a_{rs} is the smallest possible integer such that

$$ra_{rs} = s \text{ (mod. } p).$$

Taking $r = 1$ we see that $a_{1s} = s$, and that consequently the first row is

$$1, 2, 3, \ldots, \tfrac{1}{2}(p-1).$$

Similarly taking $s = r$ we have $ra_{rr} = r$, showing that each diagonal element is 1. Less easily we find the last row to be

$$p-2,\ p-4,\ p-6, \ldots, 5, 3, 1;$$

and if we try to proceed further on the same line the difficulties increase, forcing us soon to determine the elements one at a time only. The determinants for $p = 5$ and $p = 7$ are thus found to be

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 5 & 3 & 1 \end{vmatrix}.$$

It is best to begin by determining the first column, for then the s^{th} column is got therefrom by multiplying by s and 'casting out the p's'. For example, when p is 11 the elements of the first column are the lowest integers of the form $(\omega \cdot 11 + 1) \div r$, namely,

$$\begin{array}{cccccc} & \dfrac{0 \cdot 11 + 1}{1}, & \dfrac{11 + 1}{2}, & \dfrac{11 + 1}{3}, & \dfrac{11 + 1}{4}, & \dfrac{4 \cdot 11 + 1}{5} \\ \text{i.e.} & 1\quad, & 6\quad, & 4\quad, & 3\quad, & 9; \end{array}$$

then doubling these and diminishing by a multiple of 11 where possible, we have for the second column

$$2, \qquad 1, \qquad 8, \qquad 6, \qquad 7;$$

and so on, the final outcome being

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 1 & 7 & 2 & 8 \\ 4 & 8 & 1 & 5 & 9 \\ 3 & 6 & 9 & 1 & 4 \\ 9 & 7 & 5 & 3 & 1 \end{vmatrix}.$$

Similarly the first column in the next case ($p = 13$) is

$$\frac{1}{1}, \quad \frac{13 + 1}{2}, \quad \frac{2 \cdot 13 + 1}{3}, \quad \frac{3 \cdot 13 + 1}{4}, \quad \frac{3 \cdot 13 + 1}{5}, \quad \frac{5 \cdot 13 + 1}{6},$$

and the final outcome

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 1 & 8 & 2 & 9 & 3 \\ 9 & 5 & 1 & 10 & 6 & 2 \\ 10 & 7 & 4 & 1 & 11 & 8 \\ 8 & 3 & 11 & 6 & 1 & 9 \\ 11 & 9 & 7 & 5 & 3 & 1 \end{vmatrix}.$$

After performing the evaluations in these four cases and finding

$$-5, \quad 7^2, \quad 11^4, \quad -13^5$$

Malo was led to suspect the general result

$$(-p)^{\frac{1}{2}(p-3)},$$

and the rest of his paper talks helpfully of ways and means for effecting a proof.

YOUNG, A. ($1914^{22}/_1$)

[On binary forms. *Proceed. London Math. Soc.*, (2) xiii. pp. 441–495.]

Two separate passages (pp. 451–452, 460–462) of this long paper are occupied with the evaluation of determinants. One of the determinants, however, is without much difficulty seen to be included in a case of Zeipel's, $|\,(m)_p\,(m+d)_{p+e}\ldots(m+rd)_{p+re}\,|$, of 1865, namely the case where $d=1$ and $e=1$; and the other, not so easily, to be traceable to a like source, namely, the case where $d=1$ and $e=2$ (*Hist.*, iii. pp. 448–451.)

MUIR, T. ($1914^1/_4$)

[Question 17708. *Educ. Times*, lxvii. p. 206.]

A peculiar expression is here given for the number of terms in an n-line zero-axial determinant, namely,

$$(n-1)\left\{(n-2)!+\frac{2n-5}{3}(n-2)_2\cdot(n-4)!+\frac{4n-19}{5}(n-2)_4\cdot(n-6)! \right.$$
$$\left. +\frac{6n-41}{7}(n-2)_6\cdot(n-8)!+\ldots\ldots\ldots\right\}$$

where the numerators $2n - 5$, $4n - 19$, $6n - 41$, . . . are for shortness' sake written instead of $2(n - 2) - 1$, $4(n - 4) - 3$, $6(n - 6) - 5$, . . . (*Hist.*, iii. pp. 463–467).

NEUBERG, J. (1914$^{23}/_{7}$): NEVILLE, E. H. (1914/$_{8}$)

[Ueber verknüfte Determinanten vierter Ordnung. *Archiv d. Math. u. Phys.*, (3) xxiii. pp. 124–135.]

[Question 564. *Journ. Indian Math. Soc.*, vi. p. 159.]

The determinants in the first communication are

$$\begin{vmatrix} \cdot & a & b & c \\ a & \cdot & f & e \\ b & f & \cdot & d \\ c & e & d & \cdot \end{vmatrix}$$

and others differing from it in the sign of one or more elements: and what the author offers is a geometrical interpretation. In the second proof is wanted that *if* D *stands for the determinant*

$$\begin{vmatrix} 1 & m_1 & n_1 & a_1m_1 & a_1n_1 \\ 1 & m_2 & n_2 & a_2m_2 & a_2n_2 \\ 1 & m_3 & n_3 & a_3m_3 & a_3n_3 \\ 1 & m_4 & n_4 & a_4m_4 & a_4n_4 \\ 1 & m_5 & n_5 & a_5m_5 & a_5n_5 \end{vmatrix}$$

U_r, M_r, N_r, P_r, Q_r *for the cofactors of the elements of the* r^{th} *row of* D, *and* Ω_r *for* $(M_rQ_r - N_rP_r)/U_r$, *then*

$$\begin{vmatrix} \Sigma\,(a_rm_r\Omega_r) & \Sigma\,(m_r\Omega_r) \\ \Sigma\,(a_rn_r\Omega_r) & \Sigma\,(n_r\Omega_r) \end{vmatrix} = D\Sigma\,(\Omega_r).$$

COOLIDGE, J. L. (1914): LORIA, G. (1914): PASCAL, E. (1914, 1915)

[A simple algebraic paradox. *American Math. Monthly*, xxi. pp. 184–185.]

[Sur un paradoxe algébrique apparent. *American Math. Monthly*, xxi. p. 327.]

[Su di una classe di determinanti. *Rendic. . . . Accad. delle Sci. . . .* (Napoli), (3) xx. pp. 219–222: or *American Math. Monthly*, xxii. pp. 154–156.]

The initial subject here is the reconcilement of two results derived in different ways from the same set of data, namely the two-fold result

$$\left.\begin{array}{l} ac' - a'c = bd' - b'd \\ ad' + bc' = a'd + b'c \end{array}\right\}$$

and the single result

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

The want is supplied in a line or two by Loria who, without referring to Voigt's paper of 1882 (*Hist.*, iv. pp. 460, 472–473), points out that the vanishing determinant is the sum of two squares: also at greater length by Pascal, who, however, makes the same omission.

ROSEVEARE, W. N. ($1914^{17}/_{6}$): MUIR, T. ($1914^{19}/_{8}$)

[A proof . . . that every algebraic function with real coefficients has real factors of the form $x^2 - px + q$, . . . *Transac. R. Soc. S. Africa*, iv. pp. 215–221.]

[On Malet's proof that every equation has roots real or imaginary equal in number to the degree. *Transac. R. Soc. S. Africa*, iv. pp. 222–229.]

[Note on the product of a special n-line determinant by its central minor of the $(n - 4)^{\text{th}}$ order. *Transac. R. Soc. S. Africa*, iv. pp. 273–277.]

The first two papers are mainly attractive because of their treatment of the so-called "fundamental theorem of algebra": incidentally, however, they bring to notice a fresh identity in determinants which of course calls for our careful attention. The identity may be described as giving an equivalent for the product of a special n-line determinant by its central minor of the $(n - 4)^{\text{th}}$ order: for example, when n is 7 and the matrix of the determinant, $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{vmatrix}$ say, is the sum of the two matrices

$$\begin{vmatrix} a_0 & a_1 & a_2 \dots a_6 \\ . & a_0 & a_1 \dots a_5 \\ . & . & a_0 \dots a_4 \\ . & . & . \quad . \quad . \\ . & . & . \dots a_0 \end{vmatrix}, \quad \begin{vmatrix} . & \dots & . & . & -a_8\lambda \\ . & \dots & . & -a_8\lambda^2 & -a_7\lambda \\ . & \dots & -a_8\lambda^3 & -a_7\lambda^2 & -a_6\lambda \\ . & . & . & . & . \\ -a_8\lambda^7 & \dots & -a_4\lambda^3 & -a_3\lambda^2 & -a_2\lambda \end{vmatrix},$$

the equivalent for $\begin{vmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,4\,5\,6\,7 \end{vmatrix} \cdot \begin{vmatrix} 3\,4\,5 \\ 3\,4\,5 \end{vmatrix}$ is

$$\begin{vmatrix} 2\,3\,4\,5\,6 \\ 2\,3\,4\,5\,6 \end{vmatrix}^2 - \begin{vmatrix} 2\,3\,4\,5\,6 \\ 2\,3\,4\,5\,7 \end{vmatrix}^2 \lambda + \begin{vmatrix} 2\,3\,4\,5\,6 \\ 2\,3\,4\,5\,6 \end{vmatrix} \begin{vmatrix} 2\,3\,4\,5\,6 \\ 2\,3\,4\,6\,7 \end{vmatrix} \lambda.$$

In the third paper proof is given that this identity of Roseveare's is not included in Muir's theorem of 1879 (*Hist.*, iii. pp. 79–80), and an effort is made to generalize it otherwise, the outcome being

$$|\alpha_1\beta_2\gamma_3\delta_4|^2 - |\alpha_1\beta_2\gamma_3\delta_5|^2\lambda + |\alpha_1\beta_2\gamma_3\delta_4|\,|\alpha_1\beta_2\gamma_4\delta_5|\lambda$$

$$= \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_4 & . & . \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_4 & . & . \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 & . & . \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_4 & . & . \\ . & . & \delta_4 & \delta_5\lambda & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\ . & . & \gamma_4 & \gamma_5\lambda & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ . & . & \beta_4 & \beta_5\lambda & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ . & . & \alpha_4 & \alpha_5\lambda & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{vmatrix},$$

where the number of variables is more than doubled and where, but for the λ's, the determinant on the right would be centro-symmetric and therefore expressible as the product of two.

KNESER, A. (1914): KOSCHMIEDER, . (1915[16]/3)

[Zur Theorie der Determinanten. *H. A. Schwarz Festschrift*, pp. 177–191, in partic. pp. 189–191.]

[Bemerkung zur Bessel'schen Ungleichung. *Archiv d. Math. u. Phys.*, (3) xxiii. pp. 370–371.]

These may be taken as supplementary to Richardson and

Hurwitz's remarks on the specialization of their two theorems of 1909.

MUIR, T. ($1914^{28}/_{9}$)

[Properties of the determinant of an orthogonal substitution. *Proceed. R. Soc. Edinburgh*, xxxv. pp. 54–62.]

Part of this is devoted to the very special form of determinant which is accurately defined as 'the duplicant of an orthogonant', it being merely noted in addition that the said orthogonant is not necessarily specialized but has the base-index σ. Six results on the subject are formulated, the matters dealt with being evaluation and the summation of coaxial minors: for example, *The value of the duplicant of a four-line orthogonant is*

$$(\text{saxm}_2 - 2\sigma)^2:$$

and the sum of its coaxial minors of every order is

$$(\text{saxm}_2 - 2\sigma + \text{saxm}_1 + 1)^2 - 1.$$

BAILEY, W. N. ($1914^{1}/_{11}$): MARTYN, W. J. ($1915^{1}/_{1}$)

[Questions 17860, 17906. *Educ. Times*, lxvii. pp. 38, 273–274, 519: or *Math. from Educ. Times*, (2) xxviii. p. 57: xxix. p. 24.]

The first of the two results given here under 1914 is that

$$|\, a_1A_2A_3 \cdot b_2B_3B_1 \cdot c_3C_1C_2 \,| = - \,|\, a_1b_2c_3 \,| \cdot |\, a_2a_3 \cdot b_3b_1 \cdot c_1c_2 \,|^2$$

on the understanding that $|\, A_1B_2C_3 \,|$ denotes the adjugate of $|\, a_1b_2c_3 \,|$. The second had already been given by Nanson in 1902. The third result is that the equality of $a_2b_3 + a_3b_2$, $a_3b_1 + a_1b_3$, $a_1b_2 + a_2b_1$ entails the nullity of

$$\begin{vmatrix} a_1 & b_1 & a_1^2 + b_1^2 \\ a_2 & b_2 & a_2^2 + b_2^2 \\ a_3 & b_3 & a_3^2 + b_3^2 \end{vmatrix},$$

—an understatement, as the determinant is the sum of two each of which vanishes.

PASCAL, E. ($1915^6/_2$, $/_3$)

[Sui determinanti gobbi a matrici. *Rendic.* . . . *Accad. delle Sci.* . . . (Napoli), (3) xxi. pp. 40–47.]

[Determinantes pseudosimétricas cujos elementos son matrices. *Revista* . . . *Soc. Mat. Española*, Ann. iv. pp. 161–169.]

The type of determinant here dealt with is Voigt's of 1882 in the form used above under Scorza. The proof is considered to be specially direct in comparison with Baltzer's of 1887 (*Hist.*, iv. pp. 472–473). First of all there is established (pp. 42–45) a deduction from the so-called Sylvester's theorem which expresses the product of two determinants as an aggregate of like products: then (pp. 45–47) by means of Laplace's expansion-theorem Voigt's determinant is expressed as an aggregate of products of n-line minors: and finally this latter aggregate is transformed into the sum of two squares by the application of the aforesaid deduction.

MUIR, T. ($1915^{14}/_3$)

[Determinants whose elements are alternating numbers. *Messenger of Math.*, xlv. pp. 21–27.]

This note follows up Spottiswoode's similarly entitled paper of 1876 (*Hist.*, iii. pp. 487–489), and mainly concerns the multiplication-theorem, which as we have seen Spottiswoode had treated with scant care. In the first place it is now shown that Spottiswoode's form of the product gives an incorrect sign in the case of the 4th order: and then a new form is established, namely,

$$|\lambda_1\mu_2\nu_3\rho_4| \cdot |\alpha_1\beta_2\gamma_3\delta_4| = (-1)^{1+2+3} \overset{+}{\left| \begin{matrix} \Sigma\lambda\alpha & \Sigma\lambda\beta & \Sigma\lambda\gamma & \Sigma\lambda\delta \\ \Sigma\mu\alpha & \Sigma\mu\beta & \Sigma\mu\gamma & \Sigma\mu\delta \\ \Sigma\nu\alpha & \Sigma\nu\beta & \Sigma\nu\gamma & \Sigma\nu\delta \\ \Sigma\rho\alpha & \Sigma\rho\beta & \Sigma\rho\gamma & \Sigma\rho\delta \end{matrix} \right|}^{+}.$$

In the next place, after formulating a series of elementary properties of permanents whose elements are alternating numbers the author takes the permanent form of the product just obtained, and treating it as the determinant form is usually treated in the

like situation he evolves the left-hand member, and thus effects a second proof. Lastly he considers the case where the two initial factors are identical, his expression found for $|\lambda_1\mu_2\nu_3\rho_4|^2$ being at variance with Spottiswoode's.

PASCH, M. (1915/4)

[Ueber Teilbarkeit im Gebiet der Determinanten. *Archiv d. Math. u. Phys.*, (3) xxiv. pp. 220–230.]

This fully detailed paper concerns itself with the notes of F. Thaer and C. Thaer of the previous year,* the one on the divisibility of $|A_1{}^2B_2{}^2C_3{}^2|$ by $|a_1b_2c_3|^2$ and the other on the divisibility of $|A_1{}^pB_2{}^pC_3{}^p \ldots Z_n{}^p|$ by $|A_1B_2C_3 \ldots Z_n|$. The author, who keeps to determinants of the third order, is apparently unaware of any previous attention given to the matter, —A. C.'s equality of 1897 (*Hist.*, iv. p. 67), Nanson's of 1902, and the second part of Muir's paper of 1907. An interesting fresh result is

$$\begin{vmatrix} (bz+cy)P & czR & byQ \\ czR & (cx+az)Q & axP \\ byQ & axP & (ay+bx)R \end{vmatrix} = 4abc\,xyz\,PQR$$

where $P, Q, R = bz - cy, cx - az, ay - bx$.

MUIR, T. ($1915^{28}/_{12}$)

[See under this heading in Chapter on Skew Determinants for matter concerning Permanents)

MACMAHON, P. A. (1915)

[Combinatory Analysis. i. xix + 300 pp. Cambridge.]

The author's important theorem, to which we drew pointed attention under the year 1893 (*Hist.*, iv. pp. 485–486), receives here more adequate treatment at his hands. The second chapter (pp. 93–98) of the third section is devoted to it: and, what is of additional value to us, quite a number of special cases, each

* See under Compound Determinants.

with its own guiding determinant, are discussed in the sections following.

MUIR, T. ($1915^1/_7$)

[Question 18033. *Educ. Times*, lxviii. pp. 274, 461: or *Math. from Educ. Times*, (2) xxix. p. 100.]

The object of this was to recall attention to Cayley's incidentally given expressions of 1859 for the zero-axial determinants of the 4^{th} and 5^{th} orders (*Hist.*, ii. p. 469), it being feared that, notwithstanding his note of 1860 specifically devoted to such expressions (*Hist.*, iii. pp. 3–5), the law of their formation had not received sufficient consideration. The outcome seems to justify this fear, for the expression for the 6^{th} order published in reply would certainly not have satisfied Cayley. In default of a better we now give

$$-\Sigma\{(12\cdot 21)(34\cdot 45)(56\cdot 61)\}+\Sigma\{(12\cdot 21)\cdot\Sigma(34\cdot 45\cdot 56\cdot 63)\}$$
$$+\Sigma\{(12\cdot 23\cdot 31)(45\cdot 56\cdot 64)\}-\Sigma\{12\cdot 23\cdot 34\cdot 45\cdot 56\cdot 61\},$$

where the numbers of final terms in the four parts of the expression are 15, 90, 40, 120.

COMPOSTO, S. ($1916/_{1-4}$)

[Sui determinanti di numeri figurati. *Giornale di Mat.*, liv. pp. 78–92.]

It would seem that by figurate numbers the author means simply combinatory numbers or binomial-coefficients. His definition, which agrees with usage already observed (*Hist.*, iv. p. 497), is that the m^{th} figurate number of the n^{th} order is $(m+n-1)_n$, so that $(p)_q$ might not unreasonably be called the $(p-q+1)^{\text{th}}$ figurate number of order q. His contribution to the subject avowedly originates in a study of the determinant

$$\begin{vmatrix} 1 & (m)_1 & (m+1)_2 & \dots & (m+n-1)_n \\ 1 & (m+1)_1 & (m+2)_2 & \dots & (m+n)_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (m+n)_1 & (m+n+1)_2 & \dots & (m+2n-1)_n \end{vmatrix}, \text{ or } B \text{ say,}$$

which was first given in an imperfect form with the value 1 in Baltzer's textbook of 1864 (*Hist.*, iii. pp. 447–448) but soon

thereafter occurred independently to others. It is necessary to have it distinguished from

$$\begin{vmatrix} 1 & (m)_1 & (m)_2 & \cdots & (m)_{n-1} \\ 1 & (m+1)_1 & (m+1)_2 & \cdots & (m+1)_{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & (m+n-1)_1 & (m+n-1)_2 & \cdots & (m+n-1)_{n-1} \end{vmatrix},$$

which also has the value 1 (*Hist.*, iii. p. 460), as confusion has arisen between the two. The former cannot properly be specified by means of its diagonal term, but the latter can. The $(r, s)^{\text{th}}$ element of the former is

$$(m + r + s - 3)_{s-1}$$

and of the latter

$$(m + r - 1)_{s-1}.$$

An opening for making the desired generalization is found in the observed fact that any element of B outside the first row and first column is got by adding the element preceding it in its row to the element preceding it in its column, this observation naturally suggesting for investigation the whole family of determinants whose elements have

$$a_{r,\,s} = a_{r,\,s-1} + a_{r-1,\,s}$$

for their recurrent law of formation (e.g. *Hist.*, iii. pp. 102–103). Of the results obtained perhaps the most interesting is

$$\begin{vmatrix} (m)_1 & (m+1)_2 & \cdots & (m+n-1)_n \\ (m+1)_1 & (m+2)_2 & \cdots & (m+n)_n \\ \cdot & \cdot & \cdots & \cdot \\ (m+n-1)_1 & (m+n)_2 & \cdots & (m+2n-2)_n \end{vmatrix} = (m+n-1)_n,$$

where on examination it will be seen that the determinant evaluated is a primary minor of B, and that in consequence the author might have stated his new result thus: *In the determinant* B *the complementary minor of the* (n, 1)th *element is equal to the* (1, n)th *element.*

The second section of the paper (pp. 85–92) deals similarly with determinants whose elements are reciprocals of combinatory numbers. The most noteworthy result is a natural companion to that just given, each element of the one being the reciprocal of

the corresponding element of the other. It may be stated thus: *The* n*-line determinant whose* (r, s)th *element is*

$$1\Big/(m + r + s - 2)_r$$

is equal to the product of the elements of its secondary diagonal multiplied by

$$(m + n)_1\,(m + n + 1)_2 \ldots (m + 2n - 2)_{n-1}.$$

TRICOMI, F. ($1916/_{1-3}$)

[Sui determinanti il cui annullarsi esprime la condizione affinche $n + 2$ punti dello spazio ad n dimensioni siano su di una stessa ipersfera. *Giornale di Mat.*, liv. pp. 93–100.]

A generalization of the equality

$$\begin{vmatrix} x_1{}^2 + y_1{}^2 & x_1 & y_1 & 1 \\ x_2{}^2 + y_2{}^2 & x_2 & y_2 & 1 \\ x_3{}^2 + y_3{}^2 & x_3 & y_3 & 1 \\ x_4{}^2 + y_4{}^2 & x_4 & y_4 & 1 \end{vmatrix} = \begin{vmatrix} (1, 2, 3) & (1, 2, 3)' \\ (1, 2, 4) & (1, 2, 4)' \end{vmatrix}$$

where

$$(1, 2, 3) = (x_1 - x_3)(x_2 - x_3) + (y_1 - y_3)(y_2 - y_3),$$
$$(1, 2, 3)' = (y_1 - y_3)(x_2 - x_3) + (x_1 - x_3)(y_2 - y_3).$$

MUIR, T. ($1916^7/_3$)

[Question 18179. *Math. Quest. and Sol.*, iii. pp. 39–42.]

The subject here is a primary minor of the axisymmetric determinant $|(r + s - 2)_{s-1}|_n$, which has the value 1, being a particular case of the determinant B mentioned a page or so back. The theorem established is that *if the elements of the last row of the minor in question*

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \\ 1 & 6 & 21 & 56 & 126 \end{vmatrix}$$

be multiplied by α, β, γ, δ, ϵ respectively, the resulting determinant equals

$$(-1)^5 \text{ (last row as altered)} \; -5,\ 10,\ -10,\ 5,\ -1).$$

Two proofs besides the author's are given, the most concise resulting from the multiplication of the determinant by 1 in the form

$$\begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ -2 & 1 & \cdot & \cdot & \cdot \\ 3 & -3 & 1 & \cdot & \cdot \\ -4 & 6 & -4 & 1 & \cdot \\ 5 & -10 & 10 & -5 & 1 \end{vmatrix}.$$

As a corollary it is noted that when the given n-line minor has the elements of its last row multiplied by 1, 2, 3, . . ., n respectively, the resulting determinant equals n^2.

MACMAHON, P. A. (1916⁹/₃)

[The number of terms in a determinant which has some zero elements. *Proceed. London Math. Soc.*, (2) xv. pp. 318–321.]

This fresh solution is an application of the author's general determinantal theorem of 1893 (*Hist.*, iv. pp. 485–486), which he now enunciates as follows: *If* $X_1, X_2, \ldots, X_n$ *be linear functions of* $x_1, x_2, \ldots, x_n$ *the determinant of whose coefficients is* $|a_1b_2c_3\ldots l_n|$, *then the coefficient of* $x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$ *in the development of* $X_1^{\xi_1}X_2^{\xi_2}\ldots X_n^{\xi_n}$ *is the same as its coefficient in the development of*

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\ldots(1-l_nx_n)|},$$

it being understood that the denominator here is a doubly abridged symbolism to be taken to indicate that, after the multiplication of the binomials in it, products such as a_1b_2, $a_1b_2c_3$, . . . are to be replaced by $|a_1b_2|$, $|a_1b_2c_3|$, . . . From this with the help of a suggestion from Laisant (*Hist.*, iv. p. 481) he concludes that *the number of terms in* $|a_1\, b_2\ldots l_n|$ *is the coefficient of* $x_1, x_2, \ldots, x_n$ *in the expansion of*

$$1 \div |(1-a_1x_1)(1-b_2x_2)\ldots(1-l_nx_n)|$$

when each of the non-zero elements has been replaced by 1. One of his examples is the 4-line determinant which has all its elements general save for zeros in the $(1, 1)^{\text{th}}$, $(1, 2)^{\text{th}}$, $(2, 3)^{\text{th}}$ places. The first requirement is to form the determinant

$$\begin{vmatrix} \cdot & \cdot & x_3 & x_4 \\ x_1 & x_2 & \cdot & x_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} .$$

the second to ascertain the sums $\sigma_1, \sigma_2, \ldots$ of its 1-line, 2-line, ... coaxial minors, the third to expand the reciprocal of

$$1-\sigma_1+\sigma_2-\sigma_3+\sigma_4, \quad \text{i.e.} \quad 1-x_2-x_3-x_4-x_1x_3-x_1x_4+x_2x_3,$$

and this being done the coefficient of $x_1x_2x_3x_4$ is found to be 10.*

MUIR, T. (1916¹/₈): AIYANGAR, S. K. (1916–1917)

[Question 18268. *Math. Quest. and Sol.*, ii. p. 81.]

[Question 731. *Journ. Indian Math. Soc.*, viii. p. 39: ix. pp. 117–118.]

The first result given here is that the discriminant of

$$\begin{vmatrix} a_1x + a_2y + a_3z & c_1x + c_2y + c_3z \\ b_1x + b_2y + b_3z & d_1x + d_2y + d_3z \end{vmatrix}$$

$$\text{is} \quad 2\Big\{ \, |a_1b_2d_3| \, |a_1c_2d_3| - |a_1b_2c_3| \, |b_1c_2d_3| \, \Big\},$$

and the second is that in a spherical triangle

$$\begin{vmatrix} \sin a & \operatorname{cosec} a & \operatorname{cosec}^2 a & \cos A \\ \sin b & \operatorname{cosec} b & \operatorname{cosec}^2 b & \cos B \\ \sin c & \operatorname{cosec} c & \operatorname{cosec}^2 c & \cos C \end{vmatrix}$$

$$\begin{aligned} = \; & \operatorname{cosec}^2 a \quad \operatorname{cosec}^2 b \quad \operatorname{cosec}^2 c \\ & \cdot (\cos a - \cos b)(\cos b - \cos c)(\cos c - \cos a) \\ & \cdot (-1 - \cos a - \cos b - \cos c). \end{aligned}$$

* The author by following Netto in a historical paragraph is unintentionally unfair to Cunningham (*Hist.*, iii. pp. 466–467).

HATTON, J. L. S. (1916/9)

[Question 18285. *Math. Quest. and Sol.*, iii. pp. 1, 90–91.]

The equality set here for proof does not really differ from one given in Dostor's first edition (1877, p. 71), namely,

$$4\begin{vmatrix} 1 & a & a' & a^2+a'^2 \\ 1 & b & b' & b^2+b'^2 \\ 1 & c & c' & c^2+c'^2 \\ 1 & d & d' & d^2+d'^2 \end{vmatrix}^2 = -\begin{vmatrix} . & u & v & w \\ u & . & x & y \\ v & x & . & z \\ w & y & z & . \end{vmatrix}$$

where $u=(a-b)^2+(a'-b')^2, \ldots$

MÜNCH, . (1916⁴/10)

[Determinantensätze abgeleitet unter Verwendung einer complexen Grösse. *Unterrichtsblätter f. Math. u. Naturw.*, xxii. pp. 115–116.]

An example of the theorems obtained in the way mentioned we may state thus: *Any* n-*by*-3 *array is evanescent whose rows belong to one or more of the types*

$$\begin{array}{lll} \cos fa, & \cos(f-1)a, & \cos(f-2)a \\ \sin ga, & \sin(g-1)a, & \sin(g-2)a \\ \cos(h-2)a, & \cos(h-1)a, & \cos ha \\ \sin(k-2)a, & \sin(k-1)a, & \sin ka. \end{array}$$

It is also, however, a simple consequence of the operation

$$\mathrm{col}_1 + \mathrm{col}_3 = \mathrm{col}_2 \cdot 2\cos a.$$

HAYASHI, T. AND SHIBATA, K. (1917/12)

[A determinantal theorem and Clifford's theorem on n lines. *Tôhoku Math. Journ.*, xiv. pp. 1–10.]

The determinantal theorem referred to concerns determinants of the type first used by Cayley in 1854 when dealing with involution and anharmonic ratios (*Hist.*, ii. pp. 451–456). Un-

fortunately, perhaps, they are not viewed as 4-line minors of the array

$$\left\|\begin{array}{cccccccc} xy & a_1b_1 & a_2b_2 & a_3b_3 & \alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_3 & \xi\eta \\ y & b_1 & b_2 & b_3 & \beta_1 & \beta_2 & \beta_3 & \eta \\ x & a_1 & a_2 & a_3 & \alpha_1 & \alpha_2 & \alpha_3 & \xi \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right\|.$$

Otherwise they might have been conveniently specified by means of the numbers of their columns in the array, the theorem then being: *If* $|1534| = |1642| = |1723| = 0$, the equations $|8267| = 0$, $|8375| = 0$, $|8456| = 0$ are not independent. In discussing the subject advantage might also then have been taken of Cayley's and other theorems regarding the evanescence of such an array.

MUIR, T. (1918/$_1$)

[Question 18574. *Math. Quest. and Sol.*, v. Part I.]

The theorem here suggested for generalization is that if $bfg = cdh$ in the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

then $$bB = dD,\ fF = hH,\ gG = cC$$

whence $$BF/g,\ FG/b,\ GB/f = DH/c,\ HC/d,\ CD/h$$

and $$BFG = CDH.$$

OGURA, K. (1918/$_4$)

[On the Fourier constants. *Tôhoku Math. Journ.*, xiv. pp. 284–296.]

Andréieff's multiplication-theorem of integrants is here used —but still without acknowledgment—to establish an important result regarding Fourier's trigonometrical series, the determinant of the ϕ's and the determinant of the ψ's being identical,

each being equal to the $(2n+1)$-line alternant whose $(s+1)^{\text{th}}$ column is

$$1,\ \cos x_s,\ \sin x_s, \ldots, \cos nx_s,\ \sin nx_s$$

and whose value had been noted in 1879 (*Hist.*, iii. pp. 167–168).

CALDARERA, F. ($1918^{20}/_1$)

[Su taluni determinanti di forme singolarė. *Atti della Accad. Gioenia* . . . (Catania), (5) xi. 17 pp.]

As we have already seen the greater part of this longish paper is occupied with alternants and allied matters. The only other determinant considered (§§ 10–12) is one of odd order whose matrix is a magic square, the case worked out in detail being that whose elements are the first 81 integers and whose every row and column therefore has the sum $\frac{1}{2}9(1+81)$. It is readily shown that we can remove in succession the factors $\frac{1}{2}9(1+9^2)$, 10, 9^7, and have remaining as cofactor the 7-line determinant

$$\begin{vmatrix} 9 & . & . & -1 & 1 & . & . \\ 1 & 9 & . & -1 & 1 & . & . \\ 1 & . & 9 & -1 & . & 1 & . \\ . & 1 & . & 9-1 & . & 1 & . \\ . & 1 & . & -1 & 9 & . & 1 \\ . & . & 1 & -1 & . & 9 & 1 \\ . & . & 1 & -1 & . & . & 9 \end{vmatrix}.$$

This last, which is of the type dealt with in Caldarera's first paper of 1866 (*Hist.*, iii. p. 478), we find for ourselves to be equal to

$$(9-1)(9^3-1)(9^3+1)$$

so that the final desired result is known.

METZLER, W. H. ($1918/_3$)

[Note on a certain class of determinants. *American Math. Monthly*, xxv. pp. 113–115.]

The class referred to is that which under Scorza (1913) we have above described as block-skew. The author's first contri-

bution is another proof of Voigt's theorem, and then he passes on to the similar form

$$\begin{vmatrix} (A) & (B) & (C) & \ldots \\ (-B) & (A) & (D) & \ldots \\ (-C) & (-D) & (A) & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

which, he says, can by a rearrangement of rows and of columns be made to take the same form as Voigt's, namely,

$$\begin{vmatrix} (A') & (B') \\ (-B') & (A') \end{vmatrix}.$$

MUIR, T. (1918¹/₆)

[Question 18663. *Math. Quest. and Sol.*, vi. pp. 51–52.]

The first and third of the results here given are that

$$|A_1a_2a_3 \cdot B_2b_3b_1 \cdot C_3c_1c_2| = -|a_2a_3 \cdot b_3b_1 \cdot c_1c_2|^2,$$
$$|aA_1 \cdot b_2B_2 \cdot c_3C_3| = |a_1b_2c_3| \cdot \{|a_1b_2c_3|^2 - |a_2a_3 \cdot b_3b_1 \cdot c_1c_2|\}$$

on the understanding that $|A_1B_2C_3|$ denotes the adjugate of $|a_1b_2c_3|$. The second had already been made known.

MUIR, T. (1918¹⁰/₆)

[Note on the representation of the expansion of a bordered determinant. *Messenger of Math.*, xlviii. pp. 23–32.]

In connection with the main subject here two special forms come into notice. The first arises from the bordering of a non-zero determinant whose non-zero elements are units all occurring in the diagonal. For this the expression obtained is the so-called product of the two bordering arrays: for example,

$$\begin{vmatrix} 1 & \cdot & \cdot & a_1 & a_2 \\ \cdot & 1 & \cdot & b_1 & b_2 \\ \cdot & \cdot & 1 & c_1 & c_2 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdot & \cdot \\ \alpha_2 & \beta_2 & \gamma_2 & \cdot & \cdot \end{vmatrix} = (-1)^2 \left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\| \left\| \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \right\|,$$

—a fact apparently recognized by Sylvester in 1852 (*Hist.*, ii. p. 199). Another special form and its development are illustrated by the example,

$$\begin{vmatrix} U & . & . & . & . & \theta_{11} & \theta_{12} & \theta_{13} \\ . & U & . & . & . & \theta_{21} & \theta_{22} & \theta_{23} \\ . & . & U & . & . & \theta_{31} & \theta_{32} & \theta_{33} \\ . & . & . & U & . & \theta_{41} & \theta_{42} & \theta_{43} \\ . & . & . & . & U & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & V & . & . \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} & . & V & . \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} & \phi_{35} & . & . & V \end{vmatrix}$$

$$= U^5V^3 + U^4V^2(\theta_{11}\ \theta_{12}\ \theta_{13} \between \phi_{11}\ \phi_{12}\ \phi_{13})$$

$$+ U^3V\sum \begin{vmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{vmatrix} \cdot \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \end{vmatrix}$$

$$+ U^2 \begin{vmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{vmatrix} \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix}.$$

A third result is stated thus: *The number of terms in an* n-*line determinant which has an* r-*line minor with none but zero elements is* n! n! / (n − r)!

LÖWENHEIM, L. (1919)

[Gebietsdeterminanten. *Math. Annalen*, lxxix. pp. 223–236.]

As the student of algebras will readily guess, a ‘ Gebietsdeterminante ’ is not a determinant, but a something with a constitution of its own carefully defined to fill a place in the Algebra of Logic as nearly analogous as may be to that of a determinant in ordinary algebra. It is therefore to the Boolean or other such logician that the author’s corpus of six-and-twenty theorems will primarily be of serious interest.

SCHMIDT, H. ($1919^3/_6$)

[Notiz über eine besondere Klasse von Determinanten, und ihre Anwendung bei Stabilitätsstudien an Atommodellen. *Physik. Zeitschrift*, xx. pp. 446–448.]

Here we have a determinant whose curious scheme of elements is not due to an algebraist's fancy, but is the direct outcome of a serious study in atomic physics. In the first place it is allied to the type which for want of something better we have characterized by the word 'block', for example, the Puchta-Noether block circulant, the Simandl block continuant, and so forth;—that is to say, its array is such that it has necessarily to be viewed as composed of specialized sub-arrays. Thus in the particular instance we have now reached the array is made up of m^2 sub-arrays each of n rows and n columns, so that the determinant is of the $(mn)^{\text{th}}$ order. In the next place each of the said m^2 component arrays is a circulant, not necessarily connected in variables with any one of its fellows, so that the determinant is thus a function of at most m^2n variables. Only one property of the determinant is explicitly referred to, namely, its expressibility as the product of n determinants of the m^{th} order. This is shown to be provable in two different ways, (1) by using the multiplication-theorem, and (2) by viewing the determinant in connection with the binary quadric of which it is the discriminant.

For ourselves, however, it is important to point out that neither proof was in a sense necessary since by mere transposition of rows and of columns the determinant is easily changeable into another which Andreoli in his paper of 1915 had shown to have exactly the factors in question. And since the result of this transposition is sure to have other uses, it may be well for us to make an independent statement of it. In succinct form it is: *The determinant of* m² *circulant arrays of the* n*th* *order is equal to a block circulant of* n *arrays each of the* m*th* *order.* For example, when m is 2 and n is 3 and we are allowed to use on the left $C(a_1, a_2, a_3)$ not for the circulant but for its array, we have

$$\begin{vmatrix} C(a_1, & a_2, & a_3) & C(b_1, & b_2, & b_3) \\ C(c_1, & c_2, & c_3) & C(d_1, & d_2, & d_3) \end{vmatrix} = C\begin{pmatrix} a_1\, b_1 & a_2\, b_2 & a_3\, b_3 \\ c_1\, d_1, & c_2\, d_2, & c_3\, d_3 \end{pmatrix}$$

or at full length

$$\begin{vmatrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_2 & a_3 & a_1 & b_2 & b_3 & b_1 \\ a_3 & a_1 & a_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ c_2 & c_3 & c_1 & d_2 & d_3 & d_1 \\ c_3 & c_1 & c_2 & d_3 & d_1 & d_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & a_2 & b_2 & a_3 & b_3 \\ c_1 & d_1 & c_2 & d_2 & c_3 & d_3 \\ a_2 & b_2 & a_3 & b_3 & a_1 & b_1 \\ c_2 & d_2 & c_3 & d_3 & c_1 & d_1 \\ a_3 & b_3 & a_1 & b_1 & a_2 & b_2 \\ c_3 & d_3 & c_1 & d_1 & c_2 & d_2 \end{vmatrix},$$

the operation performed on the rows and the columns being in symbols:

$$\begin{pmatrix} 1 & 4 & 2 & 5 & 3 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

MUIR, T. $(1919^{22}/_{7})$

[Note on a sum of products which involves symmetrically the n^{th} roots of 1. *Transac. R. Soc. S. Africa*, viii. pp. 173–178.]

The main subject here, out of which the consideration of a special determinant arises, is

$$\sum_{s=1}^{s=n} \prod_{r=1}^{r=n} (a_{r1} + a_{r2}\omega^s + a_{r3}\omega^{2s} + \dots + a_{rn}\omega^{(n-1)s})$$

where ω is a primitive n^{th} root of 1: and as an example of the condensation effected we may give

$$\Sigma\,(a_1+a_2\omega+a_3\omega^2+a_4\omega^3)\,(\dots)\,(\dots)\,(d_1+d_2\omega+d_3\omega^2+d_4\omega^3)$$

$$= 4 \cdot \begin{array}{cccc|cccc} a_1 & a_2 & a_3 & a_4 & & & & \\ \cline{1-4} b_1 & b_4 & b_3 & b_2 & c_1 & c_2 & c_3 & c_4 \\ b_2 & b_1 & b_4 & b_3 & c_4 & c_1 & c_2 & c_3 \\ b_3 & b_2 & b_1 & b_4 & c_3 & c_4 & c_1 & c_2 \\ b_4 & b_3 & b_2 & b_1 & c_2 & c_3 & c_4 & c_1 \\ \cline{5-8} & & & & d_1 & d_4 & d_3 & d_2 \end{array}.$$

The determinant in question, when of the third order, is

$$\begin{vmatrix} a_1 + a_2\omega + a_3\omega^2 & b_1 + b_2\omega + b_3\omega^2 & c_1 + c_2\omega + c_3\omega^2 \\ d_1 + d_2\omega + d_3\omega^2 & e_1 + e_2\omega + e_3\omega^2 & f_1 + f_2\omega + f_3\omega^2 \\ g_1 + g_2\omega + g_3\omega^2 & h_1 + h_2\omega + h_3\omega^2 & k_1 + k_2\omega + k_3\omega^2 \end{vmatrix}.$$

Evidently its development must take the form

$$P + Q\omega + R\omega^2,$$

one way of obtaining the latter being to transform the determinant into an aggregate of determinants with monomial elements, in which case

$$\begin{aligned} P = \quad & |147| + |258| + |369| \\ + & |159| + |267| + |348| \\ + & |168| + |249| + |357|, \end{aligned}$$

if $|mnr|$ stand for the determinant whose columns are the m^{th}, n^{th}, r^{th} of the 3-by-9 array

$$\begin{array}{ccccccccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 & e_1 & e_2 & e_3 & f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 & h_1 & h_2 & h_3 & k_1 & k_2 & k_3. \end{array}$$

Another way is to express it as an aggregate of six terms of the form

$$(a_1 + a_2\omega + a_3\omega^2)\,(e_1 + e_2\omega + e_3\omega^2)\,(k_1 + k_2\omega + k_3\omega^2),$$

and then use on these the opening result above—a course which gives us

$$P = \begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline e_1 & e_3 & e_2 & k_1 \\ e_2 & e_1 & e_3 & k_3 \\ e_3 & e_2 & e_1 & k_2 \end{array} + \begin{array}{ccc|c} b_1 & b_2 & b_3 & \\ \hline f_1 & f_3 & f_2 & g_1 \\ f_2 & f_1 & f_3 & g_3 \\ f_3 & f_2 & f_1 & g_2 \end{array} + \begin{array}{ccc|c} c_1 & c_2 & c_3 & \\ \hline d_1 & d_3 & d_2 & h_1 \\ d_2 & d_1 & d_3 & h_3 \\ d_3 & d_2 & d_1 & h_2 \end{array}$$

$$- \begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline f_1 & f_3 & f_2 & h_1 \\ f_2 & f_1 & f_3 & h_3 \\ f_3 & f_2 & f_1 & h_2 \end{array} - \begin{array}{ccc|c} b_1 & b_2 & b_3 & \\ \hline d_1 & d_3 & d_2 & k_1 \\ d_2 & d_1 & d_3 & k_3 \\ d_3 & d_2 & d_1 & k_2 \end{array} - \begin{array}{ccc|c} c_1 & c_2 & c_3 & \\ \hline e_1 & e_3 & e_2 & g_1 \\ e_2 & e_1 & e_3 & g_3 \\ e_3 & e_2 & e_1 & g_2 \end{array}.$$

There is thus deduced a curious identity connecting six determinants and six bilinears or bipartites.

MÉTROD, G. (1919$^1/_8$)

[Question 4953. *L'Intermédiaire des Math.*, xxvi. p. 100.]

The question of the possible values of $|a_{1n}|$ when $a_{rs} = 1^{\frac{1}{2}}$ is here merely raised.

OGURA, K. (1919/$_{10}$)

[Generalization of Bessel's and Gram's inequalities in the elliptic space of infinitely many dimensions. *Tôhoku Math. Journ.*, xviii. pp. 1–22.]

The inequality here spoken of as Gram's (without any justification, it may be added) is that brought to our notice by Kowalewksi in 1909 (see above), namely:

$$\begin{vmatrix} \int_a^b y_1^2 dx & \dots & \int_a^b y_1 y_n dx \\ \cdot & \cdot & \cdot \\ \int_a^b y_n y_1 dx & \dots & \int_a^b y_n^2 dx \end{vmatrix} \geqq 0.$$

In the non-geometrical part of the paper (pp. 2–9) the fundamental advance made by the author is involved in the theorem that the same determinant is not less than a sum of n squared determinants whose elements are Fourier constants of the y's with respect to a system of orthogonal functions. The results following on this are of corresponding or even greater width, the last being a test for the linear dependence of a p-by-q array of functions.

LIST OF AUXILIARY WRITINGS

Mainly Applicational

1891. Deruyts, J. Essai d'une théorie générale des formes algébriques. vi + 156 pp. Bruxelles.

1896. Frobenius, G. Ueber Gruppencharactere. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1896. pp. 985–1021, 1343–1382.

1898. Loewy, A. Ueber bilineare Formen mit conjugirt imagi-

nären Variablen. *Nova Acta* . . . (Halle), lxxi. pp. 379–446: or *Math. Annalen*, l. pp. 557–576.

1901. Cwojdzinski, K. Distanzrelation zwischen Punkten und Geraden der Ebene sowie Punkten und Ebenen im Raume. *Archiv d. Math. u. Phys.*, (3) v. pp. 118–122; ix. pp. 8–10.

1902. Dickson, L. E. An elementary exposition of Frobenius' theory of group-characters and group-determinants. (Note references to literature on p. 25.) *Annals of Math.*, (2) iv. pp. 25–49.

1902. Sire, J. Note sur les invariants ponctuels et tangentiels. *Revue de Math. Spéc.*, xii. pp. 425–430, 454–458, 476–479.

1903. Autonne, L. Sur l'hypohermitien. *Bull. . . . Soc. math. de France*, xxxi. pp. 140–155, 268–271.

1905. Alasia, C. Sui determinanti ed i caratteri di alcuni gruppi. *Rivista di fis., mat.. . . .* Anno vii. pp. 189–221.

1906. Minetola, S. Sopra alcune proprietà delle operazioni di polare. *Giornale di Mat.*, xlv. pp. 27–47: in partic. § 2, pp. 38–47.

1907. Nicoletti, O. Sulla teoria della convergenza degli algoritmi di iterazione. (See in partic. § III. and the footnote of references.) *Annali di Mat.*, (3) xiv. pp. 1–32.

1907. Hutchinson, J. I. Hermitian forms with zero determinant. *Bull. American Math. Soc.*, (2) xiv. pp. 216–220.

1908. Cwojdzinski, K. Application des déterminants. *Mathesis*, (3) viii. pp. 11–13.

1909. Kowalewski, G. Einführung in die Determinantentheorie, §§ 117–120; §§ 191–202.

1912. Birkhoff, G. D. A determinant formula for the number of ways of colouring a map. *Annals of Math.*, xiv. pp. 42–46.

1913. Dickson, L. E. On invariants and the theory of numbers. *Colloquium Lectures, American Math. Soc.*, iv. (Madison), pp. iii + 1–110.

1913. Autonne, L. Sur les matrices hypohermitiennes et les unitaires. *Comptes rendus . . . Acad. des Sci.* (Paris), clvi. pp. 858–860.

1914. Shenton, W. F. Linear combinants of systems of binary forms, . . . *American Journ. of Math.*, xxxvii. pp. 247–271 (Note bibliography, p. 271).

1915. SPEISER, A. Gruppendeterminante und Körpersdiscriminante. *Math. Annalen*, lxxvii. pp. 546–562.

1915. MUKHOPADHYAYA, S. Parametric coefficients in the differential geometry of curves. III. Fundamental formulæ. *Bull. Calcutta Math. Soc.*, ii. No. 2, pp. 27–35.

1917. SCORZA, G. Il rango di una matrice di Riemann. *Atti . . . Accad. dei Lincei*, (5) xxvi. (2) pp. 177–182.

1918. SCHUR, I. Ueber endliche Gruppen und Hermitesche Formen. *Math. Zeitschrift*, i. pp. 184–207.

1919. VOGT, H. Réduction à une forme normale d'un système d'équations différentielles simultanées linéaires à coefficients constants. *Nouv. Annales de Math.*, (4) xix. pp. 201–209.

LIST OF AUTHORS

WHOSE WRITINGS ARE REPORTED ON

SUBJECT INDEX

TO WRITINGS ON HISTORY OF DETERMINANTS

N.B.—The volumes previously published are denoted by **A, B, C, D:** the present volume by **Z:** other writings by *Proc. R.S.E.* and *Tr. R.S.S.A.*